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CONTENTS

ABSTRACT STRUCTURE OF INEQUALITIES

A. B. Soble

INTRODUCTION

Let $f(x)$, $g(x)$, and $h(x)$ be any three single-valued real functions defined over the interval (x_1, x_2).

The reader may readily verify that h passes through all intersections (if any) of f and g in the interval, and otherwise lies between them, if and only if h is a member of the family of functions:

$$\theta f + (1-\theta)g,$$

where $0 \leqq \theta(x) \leqq 1$, $(x_1 \leqq x \leqq x_2)$.

A similar result obtains if f, g, h, and θ are functions of the two real variables x and y over a given region.

In the remainder of this section only, let $x+y$ and xy represent the l.u.b. and g.l.b., respectively, of the two functions x and y.

More abstract concepts of betweenness are as follows:

Let x, y, z be distinct functions, with y distinct from $x+z$ and from xz.

Then any of the following relations is a necessary and sufficient condition that y lie between x and z, except at their intersections.

$$y(x+z) = y = y + xz \qquad [2]$$

$$xy + yz = y = (x+y)(y+z) \qquad [3]$$

$$x + z \geqq y \geqq xz$$

$$x + yz \geqq y \geqq x(y+z). \qquad [4]$$

We note that the conditions are self-dual, since interchange of "addition" and "multiplication", and reversal of direction, leave the conditions unchanged.

Betweenness is an order relation of *three* elements. In this paper we shall consider the order relation of *two* elements, called *inequality* or *majorization*.

ONE DIMENSION

Let $f(x)$ be a single-valued real function defined for all real x.

Let $x_1 < x_2 < \ldots < x_5 < x_6$.

Suppose it is desired to find the family of real functions $g(x)$ such that in the closed intervals (x_1, x_2), (x_3, x_4), and (x_5, x_6), $g(x) \geqq f(x)$; while outside these intervals $g(x) < f(x)$.

It may be readily verified that the solution is

$$g(x) = f(x) - G(x)\prod_{i=1}^{6} (x - x_i),$$

where $G(x)$ is any real function which is positive outside the intervals, and positive or zero inside. An example is $G(x) \equiv 1$.

Extension obtains for any number of non-overlapping intervals.

TWO DIMENSIONS

Let $f(x, y)$, $F_i(x, y)$, $(i = 1, 2, 3)$, be single-valued real functions defined over the real plane.

Let $F_i(x, y) = 0$, $(i = 1, 2, 3)$, denote the boundaries of three closed regions which do not overlap.

Since the boundaries $F_i = 0$ are the same as $-F_i = 0$, we may assume that the signs are chosen so that the interiors of the regions are given by $F_i < 0$.

Suppose it is desired to find the family of real functions $g(x, y)$ such that in the closed regions $F_i(x, y) \leqq 0$, $g(x, y) \geqq f(x, y)$; while outside these regions $g(x, y) < f(x, y)$.

It may be readily verified that the solution is

$$g(x, y) = f(x, y) - G(x, y)\prod_{i=1}^{3} F_i,$$

where $G(x, y)$ is any real function which is positive outside the regions, and positive or zero inside. An example is $G(x, y) \equiv 1$.

Again, the result may be extended to any number of non-overlapping regions.

There remains the question of obtaining the equation of a region when its boundary is given by a graph, map or numerical table.

If the region is star-shaped, that is, there exists an interior point (x_0, y_0) such that every ray emanating from (x_0, y_0) intersects the boundary in one and only one finite point, the equation may be found as follows:

Let the angle of the ray be θ, and let the distance from (x_0, y_0) to the boundary be ρ.

Since the numerical value of ρ is known for $-\pi \leqq \theta \leqq \pi$, it may be expanded in a Fourier series.

Hence $\rho = \rho(\sin\theta, \cos\theta)$.

Therefore,

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = \rho\left(\frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}, \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right)$$

is the equation of the boundary.

TRANSFORMATIONS

Let $F(t)$ represent the class of real single valued functions defined for **all** t in the half-open interval $t_1 < t \leqq t_2$, and lying between the constant bounds α and β as in Figure 1.

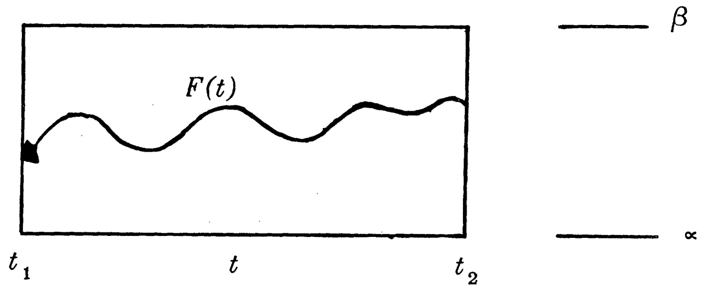

Figure 1

The transformation $\theta = \frac{\pi}{t_2-t_1}[(t-t_2)+(t-t_1)]$ and $\rho = F-\alpha$ carries $F(t)$ into a star-shaped curve about the center of, and lying within, the circle of radius $\beta-\alpha$, as in Figure 2.

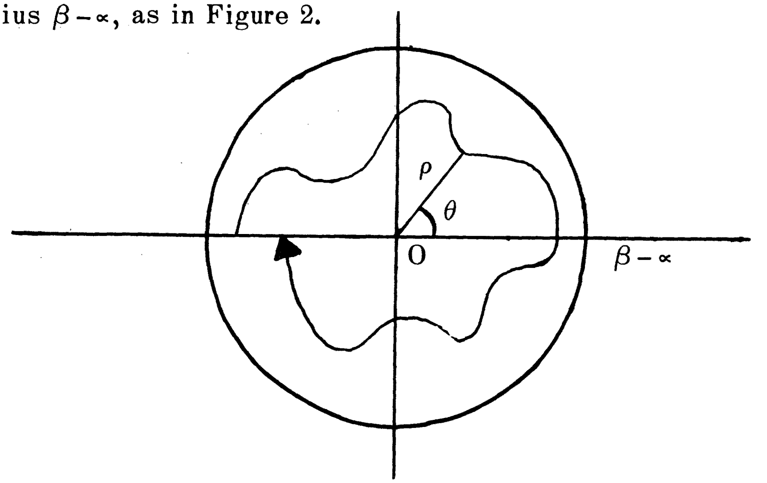

Figure 2

If $G(t) \geqq H(t)$ in the rectangle of Figure 1, then the transform of G surrounds or contains the transform of H in the circle of Figure 2, and conversely.

Let $\wp(F)$ be any monotonically increasing function of F, $\alpha \leqq F \leqq \beta$.

Let γ and δ be constants, and $\Im(t)$ a function, such that $\gamma \leqq \Im(t) \leqq \delta$, when $t_1 < t \leqq t_2$.

If $\gamma + \wp(\alpha) \geqq \alpha$ and $\delta + \wp(\beta) \leqq \beta$, then $\Im(t) + \wp(F)$ is a majorant-preserving transformation of the class of curves $F(t)$ into a sub-class.

In this paper we shall consider algebraic properties of the transformation

$$F_* = \frac{F - \alpha}{\beta - F} \quad \text{or} \quad F = \frac{\alpha + \beta F_*}{1 + F_*}$$

which converts the bounds (α, β) into $(0, \infty)$.

Since

$$\frac{\partial F_*}{\partial F} = \frac{\beta - \alpha}{(\beta - F)^2} > 0,$$

F_* increases strictly with F.

Hence the transformation preserves majorization.

We note that the transformation gives a 1 to 1 correspondence between F and F_*.

We shall also consider algebraic properties of the transformation

$$F_0 = \frac{F - \beta}{F - \alpha} \quad \text{or} \quad F = \frac{\beta - \alpha F_0}{1 - F_0}$$

which converts the bounds (α, β) into $(-\infty, 0)$.

Since

$$\frac{\partial F_0}{\partial F} = \frac{\beta - \alpha}{(F - \alpha)^2} > 0,$$

F_0 increases strictly with F.

Hence the transformation preserves majorization.

We note that the transformation gives a 1 to 1 correspondence between F and F_0.

OPERATIONS

Let $G(t)$ belong to the same class as $F(t)$.

We define the commutative binary operation $F \oplus G$ as

$$\frac{\alpha + \beta(F_* + G_*)}{1 + (F_* + G_*)}.$$

Then $(F \oplus G)_* = F_* + G_*$.

Since F_* and $G_* \geqq 0$, then $(F \oplus G)_* \geqq F_*$ and G_*.

Because of the 1 to 1 correspondence and the monotonicity, it follows that $F \oplus G \geqq F$ and G.

It remains to show that $F \oplus G$ is also of the class of F.

Now

$$(F_* + G_*) = \frac{F - \alpha}{\beta - F} + \frac{G - \alpha}{\beta - G},$$

is defined and single valued whenever F and G are.

That $F \oplus G$ is also, follows then from the definition of $F \oplus G$.

We must still show that $\alpha \leqq F \oplus G \leqq \beta$.

Since $F \oplus G \geqq F$ and G, it majorizes α.

Moreover $(F \oplus G)_* = F_* + G_* \leqq \infty$.

Hence, by the 1 to 1 correspondence and the monotonicity, $F \oplus G \leqq \beta$.

Thus $F \oplus G$ is in the original class.

We define the commutative binary operation $F \odot G$ as

$$\frac{\beta - \alpha(F_0 + G_0)}{1 - (F_0 + G_0)}.$$

Then $(F \odot G)_0 = F_0 + G_0$.

Since F_0 and $G_0 \leqq 0$, then $(F \odot G)_0 \leqq F_0$ and G_0.

Because of the 1 to 1 correspondence and the monotonicity, it follows that $F \odot G \leqq F$ and G.

It remains to show that $F \odot G$ is in the original class. This we leave to the reader.

ALGEBRA

We have already seen that the commutative binary operators $\oplus$ and $\odot$ are operators of majorization and minorization, respectively, namely

$$F \oplus G \geqq F \text{ and } G; \qquad F \odot G \leqq F \text{ and } G.$$

It may be readily verified that α and β are the identity elements of $\oplus$ and $\odot$ respectively, namely

$$\alpha \oplus F = F \qquad \text{and} \qquad \beta \odot F = F.$$

There is no inverse.

Since

$$[(F \oplus G) \oplus H]_* = F_* + G_* + H_* = [F \oplus (G \oplus H)]_*,$$

it follows that the operator $\oplus$ is associative.

Similarly the operator $\odot$ is also associative.

The two operators are not mutually distributive.

We define the complement F' of F by $\alpha + \beta - F$.

The reader may readily verify that:

$$(F')' = F, \qquad \alpha' = \beta, \qquad \text{and} \qquad \beta' = \alpha$$

$$(F')_* = -F_0 \qquad \text{and} \qquad (F')_0 = -F_*.$$

$$F_*(F')_* = F_0(F')_0 = 1 \qquad \text{and} \qquad F_0F_* = -1$$

$$(F \oplus G)' = F' \odot G' \qquad \text{and} \qquad (F \odot G)' = F' \oplus G'.$$

The algebra which we have defined we may call an *additive* algebra, because

$$(F \oplus G)_* = F_* + G_*$$

and

$$(F \odot G)_0 = F_0 + G_0 .$$

A *multiplicative* algebra may be developed by defining $(F \oplus G)^* = F^*G^*$ and $(F \odot G)^0 = F^0G^0$, under the transformations

$$F^* = \frac{\beta - \alpha}{\beta - F}, \qquad (\alpha, \beta) \to (1, \infty)$$

and

$$F^0 = \frac{F - \alpha}{\beta - \alpha}, \qquad (\alpha, \beta) \to (0, 1).$$

This we leave to the reader.

Still another algebra of majorants is that given by lattices [1], in terms of l.u.b. and g.l.b.

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THE COEFFICIENTS OF $\frac{\cosh x}{\cos x}$ AND A NOTE ON CARLITZ'S COEFFICIENTS OF $\frac{\sinh x}{\sin x}$

J. M. Gandhi

INTRODUCTION. The present paper is divided into two parts. The properties of the coefficients of $\frac{\cosh x}{\cos x}$ and of $\frac{\sinh x}{\sin x}$ will be discussed separately in two parts.

PART 1

THE COEFFICIENTS OF $\frac{\cosh x}{\cos x}$

Let

$$\frac{\cosh x}{\cos x} = \sum_{n=0}^{\infty} S_{2n} \frac{x^{2n}}{(2n)!}. \qquad [1]$$

Whence

$$\left[\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right] = \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{\infty} S_{2n} \frac{x^{2n}}{(2n)!}\right]. \qquad [2]$$

Simplifying and equating the coefficients of x^{2n} we get

$$\sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{2n}{2n-2\gamma} S_{2n-2\gamma} = 1. \qquad [3]$$

From [3] we get :

$$S_0 = 1,$$

$$\binom{2}{2} S_2 - \binom{2}{0} S_0 = 1,$$

$$\cdots$$

$$\binom{2n}{2n} S_{2n} - \binom{2n}{2n-2} S_{2n-2} + \ldots (-1)^n \binom{2n}{0} S_0 = 1.$$

Solving for S_{2n} and using Cramer's rule we get

$$S_{2n}=(-1)^n\begin{vmatrix} 1 & 1 & 0 & 0 \ldots & 0 \\ 1 & -\binom{2}{0} & \binom{2}{2} & 0 \ldots & 0 \\ 1 & \binom{4}{0} & -\binom{4}{2} & \binom{4}{4} 0 \ldots & 0 \\ 1 & \ldots & & & \\ 1 & \ldots & & & \\ 1 & (-1)^n\binom{2n}{0} & (-1)^{n-1}\binom{2n}{2} & \ldots & -\binom{2n}{2n-2} \end{vmatrix} \qquad [4]$$

TABLE

$S_0 = 1$ $S_2 = 2$ $S_4 = 12$ $S_6 = 152$

$S_8 = 3,472$ $S_{10} = 126,752$

Now we shall prove that

(A) All S's are even positive integers.

(B) The last digit of S_{2n} is 2 for $n \geqq 1$, whence $S_{2n} - S_{2n-2k} \equiv 0 \pmod{10}$ for $k < n$.

(C) $S_{4n} \equiv 0 \pmod{4n-1}$, if $4n-1$ is prime.

Theorem (C) can be easily proved from formula [3]. Proof of (A) : rewrite formula [3] as :

$$S_{2n} = \binom{2n}{2n-2}S_{2n-2} - \binom{2n}{2n-4}S_{2n-4} + \ldots - (-1)^n\binom{2n}{0}S_0 + 1. \qquad [5]$$

If S_0, S_2, ... S_{2n-2} are integers we see that S_{2n} is also an integer, but from the table we see that S_0, S_2, ... S_{2n-2} are integers for some value of n, hence they are *integers* for $n+1$, whence the theorem follows by induction.

Also if S_2, S_4, ... S_{2n-2} are even and since $(-1)^n\binom{2n}{0}S_0+1$ is 2 or zero, S_{2n} are even and the theorem that all S's are *even* follows by induction.

We shall prove [See page 5, formula 18] that $S_{2n} = [E+1]^{2n}$, where E's are Euler's numbers and all are positive whence S_{2n} are all *positive*.

Proof of (B) : rewrite formula [3] as

$$S_{2n} = \binom{2n}{2n-2}S_{2n-2} - \binom{2n}{2n-4}S_{2n-4} + \ldots + (-1)^n S_0 + 1.$$

Let us first consider the case when n is odd.

Then

$$S_{2n} = \binom{2n}{2n-2}S_{2n-2} - \binom{2n}{2n-4}S_{2n-4} + \ldots \binom{2n}{2}S_2 + 2. \qquad [6]$$

Assume that

$$S_2, S_4, S_6 \dots S_{2n-2} \equiv 2 \text{ (Mod 10)} \qquad [7]$$

Then from [6] we find that

$$S_{2n} \equiv 2 + 2[\tbinom{2n}{2n-2} - \tbinom{2n}{2n-4} + \dots \tbinom{2n}{4} - \tbinom{2n}{2}] \text{ [Mod 10]}$$

Since the value of the bracketed term is zero we get

$$S_{2n} \equiv 2 \quad \text{(Mod 10)} \qquad [8]$$

From the table we see that if [7] is true for some value of n, it is true for $n+1$, whence the result follows by induction, provided that n is odd.

Now consider the case when n is even. This case is an interesting one as here are used the properties of the imaginary quantity i, i.e. $\sqrt{-1}$.

Let $n = 2k$, formula [3] becomes

$$S_{4k} = \tbinom{4k}{4k-2}S_{4k-2} - \tbinom{4k}{4k-4}S_{4k-4} + \dots + \tbinom{4k}{2}S_2 \qquad [9]$$

Assume that :

$$S_2, S_4, \dots S_{4k-2} \equiv 2 \quad \text{(Mod 10)} \qquad [10]$$

Then from [9] we get :

$$S_{4k} \equiv 2[\tbinom{4k}{4k-2} - \tbinom{4k}{4k-4} + \dots + \tbinom{4k}{2}] \quad \text{[Mod 10]} \qquad [11]$$

By using Binomial theorem the following identity can be easily proved.

$$\tbinom{4k}{4k-2} - \tbinom{4k}{4k-4} + \dots \tbinom{4k}{4} + \tbinom{4k}{2} = 2 - \frac{(1+i)^{4n} + (1-i)^{4n}}{2} \qquad [12]$$

Since $(1+i)^2 = 2i$ and $(1-i)^2 = -2i$, formula [12] becomes :

$$\tbinom{4k}{4k-2} - \tbinom{4k}{4k-4} + \dots \tbinom{4k}{4} + \tbinom{4k}{2} = 2 - (-1)^k 2^{2k} \qquad [13]$$

Substituting [13] in [11] we get

$$S_{4k} \equiv 2[2 - (-1)^k 2^{2k}] \quad \text{[Mod 10]} \qquad [14]$$

Consider first that k is even. Let $k = 2m$. Then

$$S_{8m} \equiv 4 - 2(16)^m \quad \text{(Mod 10)}$$
$$\equiv 4 - 2\cdot 6 \equiv -8 \equiv +2 \quad \text{(Mod 10)} \qquad [15]$$

Now consider that k is odd. Let $k = 2m+1$. Then

$$S_{4(2m+1)} \equiv 4+8(16)^m \pmod{10}$$
$$\equiv 4+8\cdot 6 \equiv 2 \pmod{10} \qquad [16]$$

Now from the table we see that [10] is true for some value of k, from [15] and [16] it follows that it is true for $k+1$, whence the theorem follows by induction.

From the table we make the following conjectures:

(A) $n(2n-1)S_{2n-2} \geqq S_{2n}$

(B) $S_{2n} \equiv 0 \pmod{2^n}$.

Conjecture (B) can be proved if we can prove that k_{2n} are positive *odd* integers, where k_{2n} are given by

$$\frac{\cosh\frac{x}{\sqrt{2}}}{\cos\frac{x}{\sqrt{2}}} = \sum_{n=0}^{\infty} \frac{k_{2n}x^{2n}}{(2n)!}\,.$$

Now we shall try to construct a formula relating S's with the famous Euler numbers.

Let

$$\frac{1}{\cos x} = \sum_{n=0}^{\infty} \frac{x^{2n}E_{2n}}{(2n)!} \qquad [17]$$

With this definition of Euler's numbers, all of them are positive.

Now from [1] and [17] we get :

$$\left[\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\right]\left[\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} E_{2n}\right] = \left[\sum_{n=0}^{\infty} \frac{S_{2n}\cdot x^{2n}}{(2n)!}\right]$$

Equating the coefficients of x^{2n} and after some manipulation we get

$$S_{2n} = [E+1]^{2n}. \qquad [18]$$

Where after expansion, the index is to be replaced by the subscript.

Equation [18] gives a method to calculate S's if E's are known. Incidently [1] also implies that

$$\frac{\cos x}{\cosh x} = \sum_{n=0}^{\infty} (-1)^n S_{2n} \frac{x^{2n}}{(2n)!} \qquad [19]$$

so that from [1] and [19] we get :

$$\sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{2n}{2\gamma} S_{2n-2\gamma} S_{2\gamma} = 0. \qquad [20]$$

PART II.

A NOTE ON CARLITZ'S COEFFICIENTS OF $\frac{\sinh x}{\sin x}$

Let

$$\frac{\sinh x}{\sin x} = \sum_{n=0}^{\infty} \beta_{2n} \frac{x^{2n}}{(2n)!} \qquad [21]$$

Whence we get :

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \sum_{n=0}^{\infty} \beta_{2n} \frac{x^{2n}}{(2n)!} \qquad [22]$$

Equating the coefficients of x^{2n+1} and after some manipulation we get :

$$\sum_{\gamma=0}^{n} (-1)^{\gamma} \binom{2n+1}{2\gamma+1} \beta_{2n-2\gamma} = 1. \qquad [23]$$

As done in part 1, from formula [23], we can also obtain a determinant for β's. The determinant for β's had already appeared in MATHEMATICS MAGAZINE as a problem (See reference number 2).

Formula [23] gives an easy method to calculate β's. Professor Carlitz has calculated β's from the formula

$$\beta_{2m} = \frac{1}{(2m+1)} \sum_{S=0}^{m} (-1)^S \binom{2m+1}{2S} D_{2S} \qquad [24]$$

This formula for calculating β's is tedious, as for calculating β's we must know D's, while formula [3] directly gives the values of β's.

TABLE

$$\beta_0 = 1 \quad \beta_2 = \frac{2}{3} \quad \beta_4 = \frac{4}{3} \quad \beta_6 = \frac{104}{21}$$

$$\beta_8 = \frac{272}{9} \quad \beta_{10} = \frac{3104}{11} \quad \beta_{12} = \frac{79{,}808}{21}$$

It is evident that Professor Carlitz's table for β_6 and β_8 is wrong or misprinted.

We shall prove that :

(A) All β's are positive.

We shall prove that : [See formula 28, page 191]

$$(\beta + S)^{2m} = 2^{2m} \beta_{2m}.$$

We have already proved that all S's are positive and if β_2, β_4, ... β_{2m-2}

are positive then β_{2m} are also positive. Since $\beta_2, \beta_4, \ldots \beta_{2m-2}$ are positive for some value of m, it is true for $m+1$, whereby the theorem is established by induction.

(B) Theorem : If $4n+1$ is prime, β_{4n} will never contain a factor $4n+1$ in its denominator. (A special case of Carlitz's theorem, but here is a simple proof for it.)

For from [23] we get

$$\sum_{\gamma=0}^{2n-1} (-1)^\gamma \binom{4n+1}{2\gamma+1} \beta_{4n-2\gamma} = 0,$$

or

$$(4n+1)\beta_{4n} = -\sum_{\gamma=1}^{2n-1} \binom{4n+1}{2\gamma+1} \beta_{4n-2\gamma}. \tag{25}$$

Since when $4n+1$ is prime, the right hand side of [25] is exactly divisible by $4n+1$, and since β_{2m}, when $2m < 4n$, will not contain any factor > $(2m+1)$ in its denominator, hence the result.

From the table we make the following conjecture.

(1) Numerator of $\beta_{2n} \equiv 0 \pmod{2^n}$.

Now rewrite equation [21] in the form

$$\frac{\left(\sinh\frac{x}{2}\cdot\cosh\frac{x}{2}\right)}{\left(\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)} = \sum_{m=0}^{\infty} \beta_{2m}\frac{x^{2m}}{(2m)!}$$

Whence we get

$$\left[\sum_{m=0}^{\infty}\beta_{2m}\frac{x^{2m}}{2^{2m}(2m)!}\right]\left[\sum_{m=0}^{\infty}\frac{x^{2m}}{2^{2m}(2m)!}\right] = \left[\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{2^{2m}(2m)!}\right]\left[\sum_{m=0}^{\infty}\beta_{2m}\frac{x^{2m}}{(2m)!}\right]$$

Equating the coefficients of x^{2m} and after some manipulation we get

$$(\beta+1)^{2n} = \sum_{\gamma=0}^{n} 2^{2n-2\gamma}\beta_{2n-2\gamma}\binom{2n}{2\gamma}(-1)^\gamma. \tag{26}$$

Now we shall construct a formula relating S's with β's.

Formula [21] can be written as :

$$\frac{\left(\sinh\frac{x}{2}\cdot\cosh\frac{x}{2}\right)}{\left(\sin\frac{x}{2}\cdot\cos\frac{x}{2}\right)} = \sum_{m=0}^{\infty} \beta_{2m}\frac{x^{2m}}{(2m)!} \tag{27}$$

Using [1] and [21] we get from [27]

$$\left[\sum_{m=0}^{\infty} \beta_{2m} \frac{x^{2m}}{2^{2m}(2m)!}\right]\left[\sum_{m=0}^{\infty} S_{2m} \frac{x^{2m}}{2^{2m}(2m)!}\right] = \sum_{m=0}^{\infty} \beta_{2m} \frac{x^{2m}}{(2m)!}.$$

Equating the coefficients of x^{2m}, and after some work we get

$$(\beta + G)^{2m} = 2^{2m} \beta_{2m}. \qquad [28]$$

Where as usual after expanding, the index is to be replaced by the subscript.

It is interesting to note that similar results hold good for various hyperbolic, trigonometrical and inverse functions namely coefficients of $\frac{\tanh x}{\tan x}$, $\frac{\coth x}{\cot x}$, $\frac{\operatorname{sech} x}{\sec x}$ etc. and $\frac{\sinh^{-1} x}{\sin x}$, $\frac{\cosh^{-1} x}{\cos x}$ etc. and that the denominators in several cases possess the same properties as that of Carlitz's coefficients.

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ASME TO TRANSLATE RUSSIAN JOURNAL

A leading Russian technical journal will soon be translated into English on a regular basis by The American Society of Mechanical Engineers, it was announced recently. The effort marks the first time that the engineering society has undertaken such a project.

Under a $35,000 grant from the National Science Foundation ASME will publish the bi-monthly "Journal of Applied *Mathematics* and Mechanics."

The announcement noted that the Society had undertaken the task of translation "in an attempt to correct the present situation in which the Russians are familiar with the content of most, if not all, of our technical publications, while only a few of theirs are translated for use by the English-speaking world."

Arrangements have been made with Russian scientists, said an ASME spokesman, to secure proof sheets of the Russian journal in advance of final printing, to permit speedier translation.

In the Russian language the magazine is known as "Prikladnaya Matematika i Mekhanika", usually abbreviated as "PMM".

The magazine contains the latest theoretical and practical advances made by Russian scientists in mathematics, fluid dynamics and solid state physics. Copies will be sold, on a subscription basis, to any interested persons or groups at an annual rate of $35 for the six issues. ASME members are entitled to a 20% discount. Subscriptions may be ordered from the Order Department, The American Society of Mechanical Engineers, 29 West 39th Street, New York 18, New York.

In a statement accompanying the announcement, James N. Landis, ASME President said, "Translation of the Russian Publication 'Journal of Applied Mathematics and Mechanics,' one of the leading publications of its kind anywhere in the world, is expected to make important technical contributions to the English-speaking world. It will provide many English-speaking engineers and scientists with access to the latest U.S.S.R. developments in a truly fundamental field.

"In addition to making information available, however, the very initiation of such a project is a valuable contribution in itself. It serves to underline the universal nature of science and engineering, and to emphasize that in these fields there are no natural boundaries between nations. It is my hope that this project, together with the benefits that are sure to follow, will help to break down some of the artificial barriers that now exist, and that from such relatively modest beginnings as these, better international understanding will grow and flourish."

Professor George Herrmann of Columbia University will serve as editor of the translated magazine.

THE MIDPOINT METHOD OF NUMERICAL INTEGRATION

Preston C. Hammer

It is a noteworthy fact that while virtually all calculus texts and numerical analysis books discuss the trapezoidal method of numerical integration they ignore one which is simpler to use and generally superior to the trapezoidal—namely the midpoint rectangular method. The midpoint method also may be used with the trapezoidal method to obtain bounds for an integral in some cases. The midpoint method is an application of the Newton-Cotes "open" formulas with one point.

The trapezoidal and midpoint methods over one interval of length h are given respectively in the following equations:

1) $$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}\Big(f(x_0) + f(x_1)\Big) - \frac{f''(\xi)h^3}{12} \qquad x_0<\xi<x_1$$

2) $$\int_{x_0}^{x_1} f(x)dx = hf(x_{1/2}) + \frac{f''(\eta)h^3}{24} \qquad x_0<\eta<x_1.$$

In these equations, the second derivative of the integrand is assumed to exist on $[x_0, x_1]$. It is clear that the error in the midpoint formula is about ½ the value of the error of the trapezoidal as $x_1 \to x_0$ if f'' is continuous on $[x_0, x_1]$.However, we now prove that 2) is superior to 1) always if f does not change signs on $[x_0, x_1]$.

THEOREM: Let f be a function either concave* or convex, and continuous on $[x_0, x_1]$. Then

$$\left|\int_{x_0}^{x_1} f(x)dx - \frac{h}{2}\Big(f(x_0) + f(x_1)\Big)\right| \geq \left|\int_{x_0}^{x_1} f(x)dx - hf(x_{1/2})\right|$$

and equality holds if and only if the graph of $f(x)$ is a line or a broken line with break point at $x = x_{1/2}$.

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*A function is *concave* if any line segment joining two points on its graph lies below or on the graph.

Proof: If f'' exists, which we do not assume, and f is concave, then $f'' \leqq 0$ and if f is convex $f'' \geqq 0$. The argument for either case being the same, we will assume f is concave and since adding a constant function to f does not change the error for either formula we may assume $f > 0$ on the interval $[x_0, x_1]$.

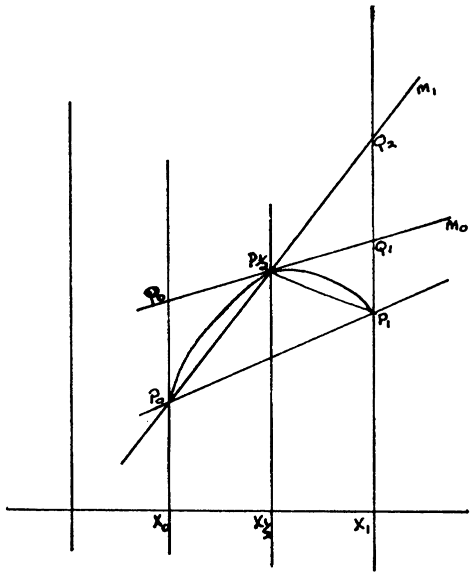

Figure 1

Consider the graph in **Figure 1.** Since f is concave, the area bounded by $y = 0$, $x = x_0$, $x = x_1$, and $y = f(x)$ is a convex set. Now the midpoint formula gives the area under *any* line through $P_{1/2}$ over $[x_0, x_1]$ (more exactly, the integral of any such linear function). Choosing m_0 as a supporting or tangent line to the curve at $P_{1/2}$ we have

$$\Delta_1 + \Delta_2 \geqq hf(x_{1/2}) - \int_{x_0}^{x_1} f(x)dx \geqq 0$$

where Δ_1 = area $P_0P_{1/2}Q_0$ and Δ_2 = area $P_{1/2}P_1Q_1$. Now, on the other hand, the trapezoidal formula gives an approximation which is less than the integral and

$$\int_{x_0}^{x_1} f(x)dx - \frac{h}{2}\Big(f(x_0) + f(x_1)\Big) \geqq \Delta_3 = \text{area } P_0P_1P_{1/2}.$$

But if we now take line m_1 through $P_0P_{1/2}$ we readily see $\Delta_4 = \Delta_1 + \Delta_2$ = area $P_1Q_2P_{1/2} = \Delta_3$. Hence

$$\int_{x_0}^{x_1} f(x)\,dx - \frac{h}{2}\Big(f(x_0)+f(x_1)\Big) \geqq hf(x_{1/2}) - \int_{x_0}^{x_1} f(x)\,dx.$$

Equality can happen only if f is linear or if the graph of f consists of two line segments joined at $P_{1/2}$. Hence, the midpoint formula is superior to the trapezoidal for either concave or convex functions and has an error opposite in sign.

COROLLARY: If f is concave and continuous on $[x_0, x_n]$ then

3) $$h\sum_{i=0}^{n-1} f(x_{i+1/2}) \geqq \int_{x_0}^{x_n} f(x)dx \geqq h\sum_{i=0}^{n} f(x_i) - \frac{h}{2}\Big(f(x_0)+f(x_1)\Big)$$

and

4) $$\left| h\sum_{i=1}^{n-1} f(x_{i+1/2}) - \int_{x_0}^{x_n} f(x)dx \right| \leqq \left| h\sum_{0}^{n} f(x_i) - \frac{h}{2}\Big(f(x_0)+f(x_1)\Big) - \int_{x_0}^{x_n} f(x)dx \right|$$

where the equality signs in the latter case hold only if the graph of f is a line or a broken line with vertices at $P_{i+1/2}$. If f is convex the inequality 4) holds and the inequality signs are reversed in 3).

Simpson's rule results if we take a weighted average of the midpoint and trapezoidal rules giving weight 2/3 to the former and 1/3 to the latter. That the midpoint rule is easier to use than the trapezoidal in general, either for hand calculation or by machine, is obvious unless there are reasons why $f(x_{i+1/2})$ is more difficult to compute. Calculating both, in case f is either convex or concave, would provide an error bound for the integral.

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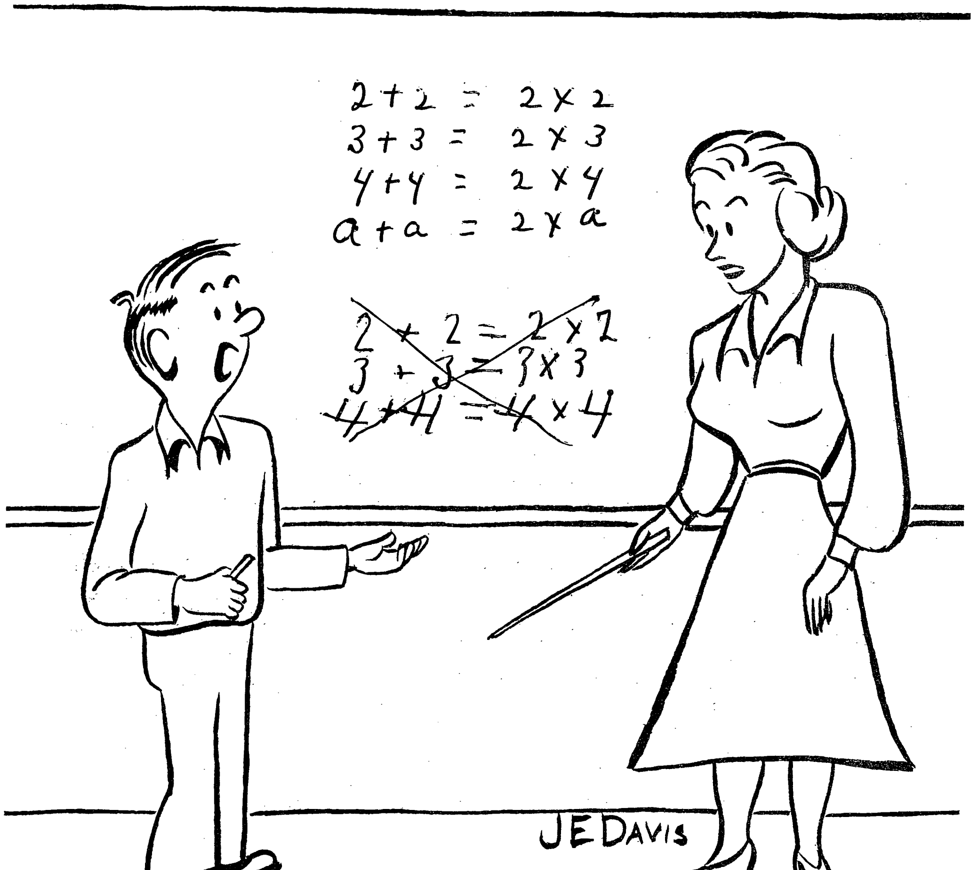

BUT, AIN'T MINE RIGHT TOO?!

A DETERMINANT FORMULA FOR HIGHER ORDER APPROXIMATION OF ROOTS

J. M. Wolfe

1. *Introduction.* Newton's method of approximation of a root of a function replaces the original function $y = f(x)$ by a linear function having the same value and the same value of the first derivative as the given function at a value of x near the root sought. In this article a convenient determinant formula is presented for the approximation when the agreement of the approximating function with the original function is extended to derivatives of higher orders at the point of contact. This formula is based on the method shown by Halley (1694) [1], Lagrange [6], Willers [11], and many others, for example, Stewart [9] and Snyder [8]. The method has been extended to n equations in n variables by Frank [4], for example.

2. *The approximating function.* Let the approximating function be

$$x = a_0 + a_1 y + a_2 y^2 + \cdots + a_{n-1} y^{n-1} + a_n y^n \tag{1}$$

where the values of the coefficients can be determined by taking $x = x_1$ and y and its derivatives equal to the corresponding values derived from the original function at the specifically selected point (x_1, y_1) near the root sought.

The simultaneous equations determining the coefficients in the approximating function are

$$\left\{\begin{array}{lllll} a_0 + & y a_1 + \cdots + & y^{n-1} a_{n-1} + & y^n a_n & = x \\ 0 + & D(y) a_1 + \cdots + & D(y^{n-1}) a_{n-1} + & D(y^n) a_n & = 1 \\ 0 + & D^2(y) a_1 + \cdots + & D^2(y^{n-1}) a_{n-1} + & D^2(y^n) a_n & = 0 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & + D^{n-1}(y) a_1 + \cdots & + D^{n-1}(y^{n-1}) a_{n-1} & + D^{n-1}(y^n) a_n & = 0 \\ 0 + & D^n(y) a_1 + \cdots + & D^n(y^{n-1}) a_{n-1} + & D^n(y^n) a_n & = 0 \end{array}\right. \tag{2}$$

where $x = x_1$, $y = y_1$, and $D^i(y^j)$ represents the value of the i-th derivative of y^j with respect to x at the specifically selected point (x_1, y_1) near the root sought.

3. *The formula.* When $y = 0$ in the approximating function (1), $x = a_0$. Thus it is necessary to solve the system of equations (2) only for a_0, whence

$$(3)\quad x = x_1 - \frac{\begin{vmatrix} y & y^2 & \cdots & y^{n-1} & y^n \\ D^2(y) & D^2(y^2) & & D^2(y^{n-1}) & D^2(y^n) \\ \vdots & & & \vdots & \vdots \\ D^{n-1}(y) & D^{n-1}(y^2) & & D^{n-1}(y^{n-1}) & D^{n-1}(y^n) \\ D^n(y) & D^n(y^2) & \cdots & D^n(y^{n-1}) & D^n(y^n) \end{vmatrix}}{\begin{vmatrix} D(y) & D(y^2) & \cdots & D(y^{n-1}) & D(y^n) \\ D^2(y) & D^2(y^2) & & D^2(y^{n-1}) & D^2(y^n) \\ \vdots & & & \vdots & \vdots \\ D^{n-1}(y) & D^{n-1}(y^2) & & D^{n-1}(y^{n-1}) & D^{n-1}(y^n) \\ D^n(y) & D^n(y^2) & \cdots & D^n(y^{n-1}) & D^n(y^n) \end{vmatrix}}$$

provided that the denominator is not zero.

The following table demonstrates the efficiency of formula (3), taking $n = 4$. It consists of three problems in which Frame [2] used a formula by Halley [1] with a correction for agreement as far as the second derivative.

Function	$y = x^3 - 2$	$y = 8x^3 - 6x + 1$	$y = x - \cos x$
Root (decimal approximation)	1.25992105	.766044	.739085
x_1	$\frac{5}{4}$	$\frac{3}{4}$	$\frac{\pi}{4}$
Error by the Newton-Raphson formula $x = x_1 - \frac{y_1}{y_1'}$	.00007895	.000623	.000451
Error by a formula by Halley $x = x_1 - \frac{y_1}{y_1' - \frac{y_1 y_1''}{2y_1'}}$	-.00000042	-.000018	.000011
Error by formula (3), taking $n = 4$	.00000000	.000002	.000000

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Brooklyn College

SYRACUSE PROFESSOR NAMED HEAD OF YESHIVA U. MATH INSTITUTE

Dr. Abe Gelbart, a Syracuse University professor, who was inspired to pursue a life-long career in mathematics through a chance meeting with the late Professor Jekuthiel Ginsburg in New York's Public Library at the age of 11, today succeeded his mentor as Director of Yeshiva University's Institute of Mathematics.

An expert in the fields of analysis and applied mathematics, and co-developer of the important theory of pseudoanalytic functions, Dr. Gelbart also will serve as editor of the University's noted scholarly journal, *Scripta Mathematica,* which Dr. Ginsburg founded and edited for twenty-five years.

Dr. Gelbart, who will hold the rank of professor of mathematics, fills a post left vacant since Dr. Ginsburg died last October at the age of 68.

Possessor of a fervent interest in mathematics from his earliest years, Dr. Gelbart first met Dr. Ginsburg in 1922 when he was 11 years old. A resident of Paterson, New Jersey, he was periodically brought into New York City by his parents for visits to the New York Public Library, at Fifth Avenue and 42nd Street. There, while each was browsing through mathematics books, he was noticed by Dr. Ginsburg.

Fascinated by the zeal of the scholarly youth, Dr. Ginsburg arranged with the boy's parents for weekly visits to his office and home in New York. "There we would engage in informal talks about a great variety of subjects, actually anything I wanted to discuss, but especially about mathematics," Dr. Gelbart recalls. "He kept me stimulated, always introducing new and exciting facets of the subject. But most important he guided my reading, insisting that I read only those works which were significant. In that way, he greatly accelerated my learning."

During subsequent summers, Dr. Ginsburg invited the youth to his cottage in the Catskills. Their chance meeting grew into a warm friendship, lasting until Ginsburg's death.

KNOCKING A CONE INTO A COCKED HAT

Daniel B. Lloyd

A few eyes, ears and horns attached to the accompanying figure (Figure 1) would readily transform it into a frightful creature indeed. But the author is content to leave it thus unadorned and merely let the reader imagine such unsavory possibilities. It is not considered good technique to discourage the reader too early in the article, much less to scare him away by the first picture. Just to dispel any vestige of naivité that "there aint no such animal", it might be of passing interest to mention that a certain well known toy manufacturer finds it profitable to turn these out by the thousands, and much to the delight of our younger generation.

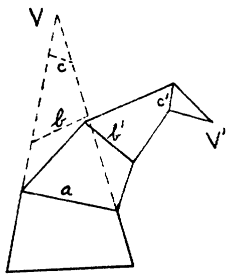

Figure 1

Of more than passing interest, however, is the mathematics involved in designing these creatures. One has a right *elliptical* cone and cuts it by a series of planes inclined to the axis of the cone at such an angle θ as to produce *circular* sections. If a rotation of 180° is then made at each joint, an accumulated bending of the axis results to produce the distortion shown. Since each joint is circular, they turn freely on each other but are not readily detachable, as the rims are designed so as to fit neatly into one another.

This property finds usefulness in the design of sheet metal flues such as are found in intake and offtake gas lines associated with smelting plants and large chemical factories. Here it is helpful to have the bend joints of circular cross-section although the flue itself may be elliptical.

We desire to find the angle that the cutting planes make with the axis of the elliptical cone, in order to cut circular cross sections. We propose to find a general expression for the angle θ in terms of α and β, where 2α is the least angle and 2β is the maximum angle at the vertex V of the right elliptical cone. Figure 2 shows front, side and top views, and also an auxiliary orthcgraphic projection of one of the oblique circular sections.

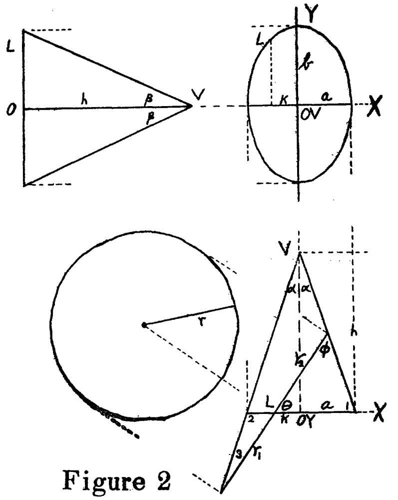

Figure 2

Referring to the side view, we have at once,

$$\frac{r_1}{\sin \angle 2} = \frac{a-k}{\sin \angle 3}$$

$$\angle 2 = 90^\circ + \alpha, \quad \angle 3 = 90^\circ - (\alpha + \theta)$$

$$\therefore r_1 = \frac{(a-k)\cos\alpha}{\cos(\alpha+\theta)}$$

Also,

$$\frac{r_2}{\sin \angle 1} = \frac{a+k}{\sin \phi}$$

$$\angle 1 = 90^\circ - \alpha, \quad \angle \phi = 90^\circ + (\alpha - \theta)$$

$$\therefore r_2 = \frac{(a+k)\cos\alpha}{\cos(\alpha-\theta)}$$

If the oblique section shown is circular, $r_1 = r_2 = r$ and

$$r_1 r_2 = r^2 = \frac{(a^2-k^2)\cos^2\alpha}{\cos(\alpha+\theta)\cos(\alpha-\theta)} \tag{1}$$

Substituting it into (2):

$$\frac{k^2}{a^2}+\frac{r^2}{b^2}=1 \qquad \text{and} \qquad r^2=\frac{(a^2-k^2)b^2}{a^2}$$

But $\quad a = h\tan\alpha, \quad$ and $\quad b = h\tan\beta \qquad (3)$

$$\therefore r^2=\frac{(a^2-k^2)\tan^2\beta}{\tan^2\alpha} \qquad (4)$$

Equating (1) and (4),

$$\frac{\cos^2\alpha}{\cos(\alpha+\theta)\cos(\alpha-\theta)}=\frac{\tan^2\beta}{\tan^2\alpha}$$

$$\frac{\cos^2\alpha}{\tfrac{1}{2}(\cos 2\alpha+\cos 2\theta)}=\frac{\tan^2\beta}{\tan^2\alpha}$$

$$\cos 2\alpha+\cos 2\theta=\frac{2\sin^2\alpha}{\tan^2\beta}$$

$$\cos 2\theta=\frac{2\sin^2\alpha}{\tan^2\beta}+2\sin^2\alpha-1$$

$$\cos^2\theta=\frac{\sin^2\alpha}{\tan^2\beta}+\sin^2\alpha=\sin^2\alpha(1+\cot^2\beta)$$

$$\therefore \cos\theta=\frac{\sin\alpha}{\sin\beta} \qquad \beta>\alpha$$

Another solution, interesting by comparison, is the following:
With V as the origin of cartesian coordinates, the cone's equation is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{h^2}=0 \qquad (5)$$

Rotating axes through the desired angle θ,

$$x = x_1\cos\theta - z_1\sin\theta$$

$$y = y_1$$

$$z = x_1\sin\theta + z_1\cos\theta$$

Then (5) becomes:

$$\left(\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{h^2}\right)x_1^2+\frac{y_1^2}{b^2}+\left(\frac{\sin^2\theta}{a^2}-\frac{\cos^2\theta}{h^2}\right)z_1^2+(\ldots)x_1z_1=0 \qquad (6)$$

For z_1 any constant ($\neq 0$), we ask that (6) shall yield circular sections.

This requires:

$$\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{h^2} = \frac{1}{b^2} \tag{7}$$

Recalling (3) above, we have finally:

$$\cos\theta = \frac{\sin\alpha}{\sin\beta} \tag{8}$$

Evidently θ increases with $\beta - \alpha$; or, the greater the eccentricity of the right section, the more exaggerated will be the resulting bend in the axis of the cone. For a right circular cone, $\theta = 0$, and no bending is possible. The results are general for α and β within the open intervals $0 < \alpha < 90°$, $0 < \beta < 90°$.

The amount of bending of the axis of the cone is 2θ for each cut, so that any desired bending may be obtained, – within limits. The size of the maximum vertex angle of the cone will determine how many cuts can be made without them overlapping.

Evidently, if $\theta > 90° - \alpha$ the section cut would be a hyperbola. The limitation $\theta < 90° - \alpha$ is also apparent from formula (8), wherein if $\theta = 90° - \alpha$, $\beta = 90°$.

More general cases of rotation, – other than 180°, lead to interesting possibilities which we shall leave to the reader's investigation. We close with the observation that the cone, as a never-failing source of interest, has been a favorite lure of mathematicians through the ages.

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NORMAL CURVE AREAS AND GEOMETRIC TRANSFORMATIONS

David Gans

It is customary to enploy the methods of the calculus in order to obtain the statistically important knowledge concerning the areas under the normal curves, i.e., the curves of the family

(I) $$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2},$$

where $\bar{x}$ and σ are parameters taking on all real values, excluding $\sigma = 0$. Viewed broadly, this knowledge can be regarded as consisting of two facts:

(1) the tabulated values of the so-called normal curve areas, i.e., the areas under the particular normal curve ($\sigma = 1$, $\bar{x} = 0$)

(II) $$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2};$$ [1]

(2) the fact that the area under an arbitrary normal curve (I) from x_1 to x_2 is precisely equal to the area under the curve (II) from t_1 to t_2, where $t_1 = (x_1 - \bar{x})/\sigma$ and $t_2 = (x_2 - \bar{x})/\sigma$.

While there would appear to be no alternative to using the methods of calculus in order to obtain the knowledge in item (1), the same cannot be said for that in item (2). In fact, as will now be shown, this knowledge can also be obtained, and perhaps more easily, by using familiar properties of the simplest geometric transformations.

Thinking of equation (I) as representing a specific curve corresponding to any fixed values of $\bar{x}$ and σ other than 0 and 1, respectively, let us subject the plane successively to the following three transformations:

(a) $x' = x - \bar{x}, \quad y' = y;$

(b) $x'' = x'/\sigma, \quad y'' = y';$

(c) $t = x'', \quad z = \sigma y''.$

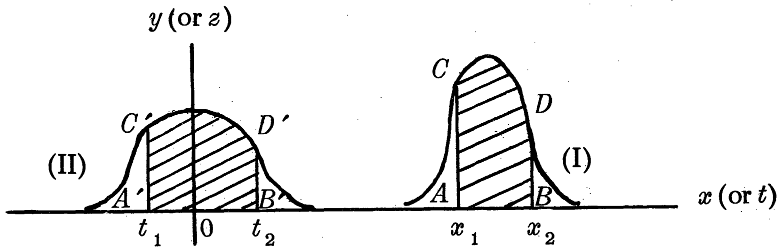
These transformations represent, respectively, a translation, a horizontal

[1]For convience we use t instead of x.

one-dimensional strain, and a vertical one-dimensional strain. The resultant, T, of these transformations, namely,

$$\text{(III)} \qquad t = (x - \bar{x})/\sigma, \qquad z = \sigma y,$$

is a continuous (affine) transformation which, as is easily verified, sends the curve (I) into the curve (II). Also, T transforms the locus $y=0$ into the locus $z=0$, and $x=$ a constant k into $t=$ the constant $(k-\bar{x})/\sigma$, that is, the horizontal axis goes into itself and vertical lines go into vertical lines. Since the transformations (a), (b), (c), respectively, preserve area, multiply area by $1/\sigma$, and multiply area by σ, the transformation T preserves area.



It follows that the figure $ABCD$, where CD is any arc of (I), AC, BD are the ordinates of C, D, and AB is a segment of the horizontal axis, will be transformed by T into the figure $A'B'C'D'$, where $C'D'$ is the corresponding arc of (II), $A'C'$, $B'D'$ are the ordinates of C', D', and $A'B'$ is a segment of the horizontal axis, and hence that the areas enclosed by the two figures are equal. If the abscissas of C, D are x_1, x_2 and those of C', D' are t_1, t_2, then, according to (III), $t_1 = (x_1 - \bar{x})/\sigma$ and $t_2 = (x_2 - \bar{x})/\sigma$.

New York University

COMBINATORIAL DERIVATIONS OF TWO IDENTITIES

Donald A. Steinberg

ABSTRACT. This paper shows, by means of two examples, the possibilities of deriving expressions by the use of the notions of permutation and combination. One of the examples is the binomial theorem.

By the expansion of various polynomial expressions it is possible to derive many identities in integers, some having practical applications, and others exemplifying curious properties of numbers. This method is utilized in Netto's *Lehrbuch der Combinatorik* to obtain a large number of such identities. Many of these may be derived using different techniques. In this paper I propose to use the notions of permutation and combination to derive a familiar identity, the binomial theorem, and also to derive a new identity which, like the binomial theorem, is an expansion of a quantity raised to a power.

1. We first see how the notions of permutation and combination can be used to give a simple proof for the binomial theorem. Consider a sequence $x_1, x_2, \cdots, x_n$ of n variable terms and a domain $\{a_1, \cdots, a_m\}$ of m distinct values such that each term of the sequence can assume any of the values of the domain. Clearly the total number of actual sequences (i.e., an actual sequence being obtained by giving each x_i a value from $\{a_1, \cdots, a_m\}$) is m^n. We can also speak of this as the total number of arrangements of the sequence. We now partition $\{a_1, \cdots, a_m\}$ into two subsets A and its compliment A', and specify that neither subset be empty. Let the number of elements in A be p and the number in A' be $(m-p)$. Let us consider the number of arrangements in which values from A are assumed by any k terms of the sequence. The expression for this is easily seen to be:

$$\binom{n}{k} p^k .$$

The remaining $(m-p)$ values are assumed by the remaining $(n-k)$ terms in $(m-p)^{n-k}$ ways. Thus the total number of ways in which any k terms can assume p values is:

$$\binom{n}{k} p^k (m-p)^{n-k} .$$

If we let $k=0, 1, \cdots, n$ every possible arrangement is counted at least once. For, every arrangement can be characterized by the number and positions of terms the values from A occupy. No arrangement is counted more than once, for that would mean values from A occupy k and k' terms in the same arrangement, where $k \neq k'$. Hence:

$$\sum_{k=0}^{n} \binom{n}{k} p^k (m-p)^{n-k} = m^n .$$

Putting $m-p = q$, $m = p+q$, we have:

$$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (q+p)^n ,$$

which is the desired result.

2. We now employ a different combinatorial situation to derive a new identity. Again we consider a sequence of length n and a domain of m distinct values. We distinguish two cases, that for which $m \leqq n$ and that for which $m > n$.

First let $m \leqq n$. We require that the first k terms of the sequence assume distinct values and that the $(k+1)$ term assume a value already assumed by one of the first k terms. The remaining $(n-k-1)$ terms have no restrictions. The number of such arrangements is seen to be:

$$\frac{m!\, k m^{n-k-1}}{(m-k)!}.$$

Let $k=1, 2, \cdots, m$. Then every possible arrangement will be counted once and only once. Every arrangement will be counted at least once because every arrangement has the property that for some k, its first k and not more than k terms are distinct. Conversely, no arrangement will be counted more than once; for, no arrangement has the property that for k and k', *its first* k *and not more than* k, *and its first* k' *and not more than* k' are distinct where $k \neq k'$. The number m is the proper upper limit of the summation. For, since the domain consists of m values we can have arrangements in which there are at most m distinct values. Thus we have:

$$\sum_{k=1}^{m} \frac{m!\, k m^{n-k-1}}{(m-k)!} = m^n$$

or

$$\sum_{k=1}^{m} \frac{k}{(m-k)!\, m^{k+1}} = \frac{1}{m!}.$$

If $m > n$ the preceding discussion remains valid except for that which is concerned with the upper limit. In this case n becomes the upper limit because, since there are n terms in the sequence, there can be at most n distinct values. Further, we note that when $k = n$, the number of arrangements is not:

$$\frac{nm!}{m\,(m-n)!},$$

which is the result of putting $k = n$ in the general expression, but is:

$$\frac{m!}{(m-n)!}.$$

Our identity now is:

$$\sum_{k=1}^{n-1} \frac{km!\, m^{n-k-1}}{(m-k)!} + \frac{m!}{(m-n)!} = m^n.$$

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NOTE ON THE DERIVATIVES OF THE LEGENDRE POLYNOMIALS

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As is well known, see [1], the Legendre Polynomials $P_n(x)$ in the interval $-1 \leqq x \leqq 1$ assume their maximum absolute value at $x = 1$ i.e. $|P_n(x)| \leqq P_n(1) = 1$. Furthermore for $n > 0$ and x interior to the interval, $|P_n(x)| < P_n(1)$. We shall show that these properties also hold for all the derivatives of the Legendre Polynomials.

Theorem

If
$$P_n^{(m)}(x) = \frac{d^m P_n(x)}{dx^m}$$
then for $0 \leqq m \leqq n$
$$|P_n^{(m)}(x)| \leqq P_n^{(m)}(1) \qquad -1 \leqq x \leqq 1 \tag{1}$$

Remark : Actually, it is known, see [2], that
$$P_n^{(m)}(1) = \frac{(n+m)!}{(n-m)!\, m!\, 2^m}$$
for all m and n.

Proof

For $n = 0$ the theorem is correct, as $P_0^{(0)}(x) = P_0^{(0)}(1) = 1$. Suppose the theorem is correct for $n-1$ and all $m \leqq n-1$ and we shall prove it for n. For $m = 0$ it is the above statement about the Legendre Polynomials. Using the recursion formula, see [2],
$$P_n^{(m)}(x) = (n+m-1)P_{n-1}^{(m-1)}(x) + xP_{n-1}^{(m)}(x) \tag{2}$$
we have for $0 < m < n$
$$|P_n^{(m)}(x)| \leqq (n+m-1)|P_{n-1}^{(m-1)}(x)| + |xP_{n-1}^{(m)}(x)|$$
$$\leqq (n+m-1)P_{n-1}^{(m-1)}(1) + P_{n-1}^{(m)}(1) = P_n^{(m)}(1)$$
For $m = n$ we have $P_n^{(n)}(x) = P_n^{(n)}(1)$ which concludes the proof.

Furthermore it follows from (2) that for $m < n$ the inequality (1) holds as a strict inequality for any point x inside the interval $-1 < x < 1$ i.e. $|P_n^{(m)}(x)| < P_n^{(m)}(1)$.

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A METHOD FOR FINDING THE REAL ROOTS OF CUBIC EQUATIONS BY USING THE SLIDE RULE

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Methods for solving cubic equations by means of the slide rule have been investigated by various authors, [1] and [2]. This note will give a somewhat different treatment including criteria concerning the nature of the roots which are obtained in a simple manner from the coefficients of the equation.

Suppose that we wish to employ the slide rule to solve the cubic equation

$$y^3 + Ay^2 + By + C = 0. \tag{1}$$

Substituting $y = x - A/3$ into equation (1), we obtain

$$x^3 + qx + p = 0, \tag{2}$$

where

$$q = (B - 1/3\,A^2) \quad \text{and} \quad p = (2/27\,A^3 - 1/3\,AB + C). \tag{3}$$

If we denote by R_1, R_2, R_3 and r_1, r_2, r_3 the roots of equations (1) and (2) respectively, we then have

$$R_1 = r_1 - A/3, \quad R_2 = r_2 - A/3 \quad \text{and} \quad R_3 = r_3 - A/3. \tag{4}$$

Thus, our task is to first find the real roots of equation (2) by using the slide rule, and then use equations (4) to find the roots of equation (1). We shall solve equation (2) by considering the following cases.

CASE 1. *When q is a positive number, c, and p is a negative number, $-d$, in the special cubic equation* (2).

For this case, we may write equation (2) as

$$x(x^2 + c) = d, \qquad (c > 0,\ d > 0). \tag{5}$$

Cubic equations of the form (5) have only one positive real root, say, r_1. The following theorem gives us useful information concerning the nature of this root.

THEOREM 1.

(i) *If $d \leqq 1$, then $r_1 < 1$. Also $r_1 < \sqrt[3]{d}$.*

(ii) *If* $c \leqq 1$, *and* $d \geqq 2$, *then* $r_1 \geqq 1$. *Also* $r_1 < \sqrt[3]{d}$.

(iii) *If* $d \geqq c+1$, *then* $r_1 \geqq 1$. *Also*, $r_1 < \sqrt[3]{d}$.

(iv) *If* $c > 1$ *and* $d < c+1$, *then* $r_1 < 1$. *Also* $r_1 < \sqrt[3]{d}$.

Proof. (i) Since $r_1 > 0$ is a root of equation (5), we have $r_1(r_1^2 + c) = d \leqq 1$ Since $c > 0$, then $r_1 < 1$, for, if $r_1 \geqq 1$, then $r_1(r_1^2 + c) > 1$, a contradiction. Since $r_1 c > 0$, then $r_1(r_1^2 + c) = d$ implies that $r_1^3 < d$. Hence $r_1 < \sqrt[3]{d}$.

(ii) Since $d \geqq 2$, then $r_1(r_1^2 + c) \geqq 2$. Since $0 < c \leqq 1$ and $r_1 > 0$, then $r_1 \geqq 1$, for, if $r_1 < 1$, then $r_1(r_1^2 + c) < 2$, a contradiction. The proof that $r_1 < \sqrt[3]{d}$ in (ii), (iii), (iv) is the same as given in (i).

(iii) Since $d \geqq c+1$, then $r_1(r_1^2 + c) \geqq c+1$. Hence, $(r_1 - 1)(r_1^2 + r_1 + c + 1) \geqq 0$. Since $r_1 > 0$ and $c > 0$, then $r_1 \geqq 1$, for, if $r_1 < 1$, $(r_1 - 1)(r_1^2 + r_1 + c + 1) < 0$, a contradiction.

(iv) Since $d < c+1$, then $r_1(r_1^2 + c) < c+1$. Hence, $(r_1 - 1)(r_1^2 + r_1 + c + 1) < 0$. Since $r_1 > 0$ and $c > 0$, then $r_1 < 1$, for, if $r_1 \geqq 1$, then $(r_1 - 1)(r_1^2 + r_1 + c + 1) \geqq 0$, a contradiction.

For this case, we use the following procedure to find the only positive real root on the slide rule.

PROCEDURE 1.

(1) Move the hairline to the number d on the D scale.

(2) Move the slide until the value of x on the C scale is under the hairline and the left (or right) index of the C scale is on the value of $(x^2 + c)$ on the D scale.

(3) The real positive root, $x = r_1$, is read under the hairline on the C scale.

REMARK 1. *When q is a positive number c, and p is a positive number d, in equation* (2), we may also use CASE 1 by simply finding the real root of $f(-x) = 0$, where $f(x) = x^3 + cx - d$.

CASE 2. *When q is a negative number, $-c$, and p is a negative number, $-d$, in the special cubic equation* (2).

For this case, we may write equation (2) as

$$x(x^2 - c) = d, \qquad (c > 0,\ d > 0). \tag{6}$$

Cubic equations of the form (6) have always one real positive root, say, r_1. The following two essential theorems can be easily proven.

THEOREM 2. *If r_1 is the real positive root of equation* (6), *then r_2 and r_3 are real negative roots of the same equation if and only if $r_1^2 \leqq 4/3\, c$.*

THEOREM 3. (i) *If r_1 is the real positive root of equation* (6), *then $r_1 > \sqrt{c}$.*

(ii) *If* r_1 *is the real positive root of equation* (6) *and* $r_1^2 \leqq 4/3\,c$, *then* $|r_2| < \sqrt{c}$ *and* $|r_3| < \sqrt{c}$.

REMARK 2. For this case, we may check our roots by utilizing the following facts:

(i) The sum of the roots is equal to 0, that is, $r_1 + r_2 + r_3 = 0$.

(ii) The product of the roots is equal to d, that is, $r_1 r_2 r_3 = d$.

We shall use the following procedure to find the real positive root of equation (6).

PROCEDURE 2.

(1) Move the hairline to the number d on the C scale.

(2) Move the slide until the value of x on the C scale is under the hairline and the left (or right) index of the C scale is on the value of $(x^2 - c)$ on the D scale.

(3) The real positive root, $x = r_1$, is read under the hairline on the C scale.

Since the expression $(x^2 - c)$ is not altered by putting x equal to $-x$, we use Procedure 2 also to find the real negative roots of equation (6).

REMARK 3. *When* q *is a negative number,* $-c$, *and* p *is a positive number,* d, *in equation* (2), we may also use CASE 2 by simply finding the roots of $f(-x) = 0$, where $f(x) = x^3 - cx - d$.

EXAMPLE. Solve $y^3 + 3y^2 - 4.75y - 10.5 = 0$.

Using (3), we find $q = -7.75$ and $p = -3.75$. Hence, (2) becomes $x^3 - 7.75x - 3.75 = 0$ or $x(x^2 - 7.75) = 3.75$, which belongs to CASE 2. By Theorem 3, $r_1 > \sqrt{7.75} = 2.78$. From Procedure 2, we have: Move the hairline to 3.75 on the D scale. Move the slide until 3 on the C scale is under the hairline and the left index of the C scale is on 1.25 on the D scale. Hence $r_1 = 3$, since when $x = 3$, $x^2 - 7.75 = 1.25$ which is the same value found at the left index of the C scale on the D scale. From Theorem 2, since $9 \leqq 4/3\,(7.75) = 31/3$, r_2 and r_3 are real negative roots. From Theorem 3(ii), $|r_2| < 2.78$ and $|r_3| < 2.78$. Always keeping the hairline on 3.75 on the D scale, we use Procedure 2 twice to find r_2 and r_3. Move the slide until 2.5 on the C scale is under the hairline and the left index on the C scale is on 1.5 on the D scale. Hence $r_2 = -2.5$. Similarly we find $r_3 = -0.5$. From Remark 2, we see that $3 - 2.5 - 0.5 = 0$ and $3(-2.5)(-0.5) = 3.75$. Finally, using (4) we find $R_1 = 2$, $R_2 = -3.5$ and $R_3 = -1.5$.

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LINEAR DIOPHANTINE EQUATIONS

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We shall describe elementary methods for solving linear Diophantine equations, either single equations or systems of equations.

The equations with which we shall deal have integral coefficients and require integral solutions. Such equations are named after Diophantus of Alexandria, who lived probably between 100 and 300 A. D.

The methods are explained with the help of specific examples. We begin with the case of a single equation.

Method I If an equation has an unknown with coefficient 1 or -1, then all the other unknowns can be taken as arbitrary integers, and the particular unknown is then determined from the given equation.

Example 1. $4x+7y-z+3w = 18$. The solution is: Take x, y, w as arbitrary integers, and determine z from the given equation. It is clear that all solutions in integers are thus obtained.

Method II If the numerically smallest coefficient, say a, is not 1 or -1, but is a factor of each of the remaining coefficients, and is not a factor of the right member of the equation (the "constant" term), the equation is incompatible, that is, it has no solution in integers. If a divides all the coefficients and also the right member, we divide through by a, thus obtaining an equivalent equation to which Method I can be applied.

Example 2. $8x+14y-2z+6w = 36$. If we divide by 2 we get the equation of Example 1 and can apply Method I. If the right hand side were 37 instead of 36, the equation would have no solution in integers, as 37 is not divisible by 2.

Method II can also be applied to any single equation of a system of simultaneous equations. If one equation is thereby shown to be incompatible in integers, then the system is incompatible in integers.

We now continue with the case of a single equation.

Method III. If the numerically smallest coefficient, say a, is not a factor of all of the remaining coefficients, the equation can be replaced by a new equation with a non-zero coefficient numerically smaller than a, by the method used in the following example.

Example 3. $16x+42y-21z = 100$. (1)

We first set

$$x = x'-3y+z. \tag{2}$$

The coefficients on the right in (2) are determined as follows. The coefficient of x' is 1, which is the rule for the case that x has the numerically smallest coefficient in (1). The coefficients −3 and 1 of y and z respectively in (2) were chosen so as to minimize the numerical values of the coefficients of y and z in the equation obtained by substituting from (2) into (1). That equation is

$$16x' - 6y - 5z = 100. \tag{3}$$

After solving (3) in integers, we shall use (2) to determine x.

Applying Method III to (3), we set

$$z = z' - y + 3x', \tag{4}$$

which converts (3) to

$$x' - y - 5z' = 100 \tag{5}$$

which has solution

$$x',\ z' \text{ arbitrary integers,} \tag{6}$$

$$y = x' - 5z' - 100. \tag{7}$$

Substituting into (4) we get

$$z = 2x' + 6z' + 100. \tag{8}$$

Then, from (2), (7) and (8) we have

$$x = 21z' + 400. \tag{9}$$

All solutions of (1) in integers are given by (6), (7), (8) and (9).

Application of Method III, repeated as often as necessary, must eventually lead to an equation all of whose coefficients are integral multiples of a particular coefficient a. We then apply Method I or Method II, depending on whether or not $a = 1$.

For an equation in two unknowns, Method III, combined with I and II, is roughly equivalent to the standard treatment of a Diophantine equation in two unknowns.* When the number of unknowns is greater than 2, substitutions such as (2) above bring the problem to solution more quickly than the standard methods.

We turn now to systems of linear Diophantine equations, giving procedures simpler and more efficient than familiar ones.

Method IV. If a coefficient 1 or −1 appears in an equation of the system, the system can be replaced by a system in one less equation and in one less unknown. (The number of unknowns that actually appear with non-zero

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*See Uspensky and Heaslitt, *Elementary Number Theory*, New York 1939, pp. 52-64.

coefficients may, however, decrease by more than unity.) We illustrate with the system

$$3x+2y-5z+4w = 4 \tag{10}$$
$$4x-\ y+7z \qquad = 10 \tag{11}$$
$$5x+6y \qquad -2w = 9. \tag{12}$$

Noting coefficient −1 for y in (11), we add twice the left member of (11) to that of (10), and similarly on the right, obtaining

$$11x+9z+4w = 24. \tag{13}$$

Similarly, from (11) and (12) we get

$$29x+42z-2w = 69. \tag{14}$$

The given system of 3 equations in 4 unknowns is now seen to be equivalent to (11), (13), (14). But as (11) can be used to determine y when x, z and w have integral values satisfying (13) and (14), we thus see that the problem is essentially reduced to that of solving the system (13), (14), a system of 2 equations in 3 unknowns.

Method V. If no coefficient is 1 or −1, but the coefficients of some unknown are relatively prime, the system can be replaced by one having a coefficient 1 or −1. We illustrate with the system

$$6x+\cdots = a \tag{15}$$
$$15x+\cdots = b \tag{16}$$
$$20x+\cdots = c. \tag{17}$$

By adding an integral multiple of each side of an equation having numerically smallest coefficient, here 6, of the given unknown, here x, to the corresponding side of one of the other equations, we get an equation with coefficient of x numerically less than 6. Thus, multiplying (15) through by −2 and adding to (16) we get

$$3x+\cdots = b-2a. \tag{18}$$

We could similarly work with (15) and (17), but we note that (15), (16), (17) are equivalent to (15), (17), (18), and by multiplying (18) through by −7 and adding to (17) we get an equation with coefficient −1, namely

$$-x+\cdots = c-7b+14a. \tag{19}$$

The given system (15), (16), (17) is equivalent to (15), (18), (19).

If a system has no coefficient 1 or −1, and the coefficients of each unknown have some common factor, while the coefficients in each equation have no common factor, then none of Methods II, IV, V can be applied. In such a case we use

Method VI. Under the circumstances described in the preceding paragraph, we apply Method III to an equation of the system having a non-zero coefficient numerically as small as possible, but we make the substitution, such as (2), in all the equations of the system. For example, given the system

$$4x + 9y + 10z - 2w = 9 \tag{20}$$
$$20x + 6y - 5z + 14w = 7 \tag{21}$$
$$6x + 18y + 5z - 4w = 10 \tag{22}$$

we note that -2 is the numerically smallest coefficient and accordingly make the following substitution suggested by examination of equation (20)*

$$w = w' + 4y. \tag{23}$$

This results in the equations

$$4x + y + 10z - 2w' = 9 \tag{24}$$
$$20x + 62y - 5z + 14w' = 7 \tag{25}$$
$$6x + 2y + 5z - 4w' = 10, \tag{26}$$

a system having a coefficient unity. The problem of solving (20), (21), (22), is now reduced to the problem of solving (24), (25), (26), since if x, y, z, w' satisfy the latter system, w can be obtained from (23), and (20), (21), (22) will be satisfied. All solutions in integers are obtained, since if x, y, z, w satisfy (20), (21), (22) then x, y, z, w' satisfy (24), (25), (26).

In general, after getting a new system such as (24), (25), (26), we apply Method II, IV or V if possible (and here of course Method IV would be used next); otherwise continue with Method VI.

It is clear that use of Methods I to VI will result in obtaining all solutions in integers of the given system of equations, and reveal incompatibility, if there is such, when an equation is reached having a common factor of the coefficients on the left which is not a factor of the number on the right side of the equation.

Examination of Methods I through VI shows that arbitrary integers are introduced only when the work is at a stage where a single equation is to be solved, and when Method I is applied. The number of arbitrary integers is *one less than* the total number of unknowns that must satisfy the single equation if not all the coefficients are zero, and is *the same as* the number of unknowns if all the coefficients are zero (that is, the equation in question is "$0 = 0$"). In either case, the number of arbitrary integers equals the number of unknowns minus the rank of the matrix of coefficients, for the single equation. Now, examination of Methods II through VI shows

*We might also use $w = w' + 2x + 4y + 5z$, thus replacing (20) by a much simpler equation, but (21) and (22) would not be simplified.

that in each case an equation, or a system of equations, is replaced by an equation or system for which the quantity "number of unknowns minus rank of the matrix of coefficients" is unchanged. Hence the following theorem, except possibly the final sentence, is valid.

THEOREM. *If a linear equation or a system of linear equations with integral coefficients is compatible in integers, all solutions in integers are given by equating the unknowns to linear functions, with integral coefficients, of $n-r$ arbitrary integers, where n is the number of unknowns and r is the rank of the matrix of coefficients. No fewer than $n-r$ arbitrary integers will suffice.*

We now prove the final statement of the theorem. Let (S) denote the given system of equations and $t_1, t_2, \cdots, t_k$ the set of arbitrary integers on which the solutions in integers of (S) depend, so that $k = n-r$. Suppose that all solutions in integers of (S) could be given by

$$x_j = c_{1j}u_1 + c_{2j}u_2 + \cdots + c_{mj}u_m + d_j \qquad (j=1,2,\cdots,n) \tag{27}$$

with $u_1, u_2, \cdots, u_m$ integers, and $m<k$. Now the formulas for $x_1, \cdots, x_n$ in terms of $t_1, \cdots, t_k$ are the result of a sequence of formulas similar in nature to (2), that is, the new unknown is always introduced on the right with coefficient unity. We consider (2) to give formulas for x, y, z in terms of x', y, z, so that the equations $y=y$, $z=z$ may be introduced to complete the set of formulas for the change. Thus (2) with the two additional equations can be solved, giving $x'=x+3y-z$, $y=y$, $z=z$, that is, the new unknowns in terms of the old unknowns. A similar statement can be made in the general case. Since a similar situation holds at each introduction of a new set of unknowns (where of course n-1 of the equations are simply of the form "unknown equals itself"), we infer that the final set of unknowns introduced are linear functions of $x_1, \cdots, x_n$. In particular, $t_1, t_2, \cdots, t_k$ are linear functions of $x_1, \cdots, x_n$,

$$t_j = f_{1j}x_1 + \cdots + f_{nj}x_n \qquad (j=1,2,\cdots,k) \tag{28}$$

a fact which is not altered by the relationships which the final set of unknowns (namely $t_1, \cdots, t_k$ and n-k other unknowns) must satisfy in order to make $x_1, \cdots, x_n$ satisfy (S). From (27) and (28) we get

$$t_j = g_{1j}u_1 + \cdots + g_{mj}u_m + h_j \qquad (j=1,\cdots,k) \tag{29}$$

where the g's and h's are constant integers, with (29) satisfied by (t) and (u) whenever both give the same solution (x) of (S) in integers.

If the u's are allowed to take on all real values, then (29) considered as parametric equations of a locus in $t_1, \cdots, t_k$ euclidean space is a hyperplane of dimension at most $m<k$. Then all k-tuples of values of $t_1, \cdots, t_k$ corresponding to solutions of (S) in integers would be confined to this hyperplane, which is impossible since $t_1, \cdots, t_k$ can be arbitrary integers

and hence determine points not confined to any plane of dimension less than k in k-space. We infer that the theorem is true.

Conditions for compatibility of (S) can be found in a paper by Oswald Veblen and Philip Franklin.*

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*"On Matrices whose Elements are Integers", Annals of Math. (2), Vol. 23 (1921), pp. 1-15; also as Appendix II of Veblen's "Analysis Situs", second edition.

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ARITHMETICAL CONGRUENCES WITH PRACTICAL APPLICATIONS

(Popular Article)

Carol Law

When we say that something is congruent, we mean that it is in agreement or in harmony with something else. In geometry we speak of congruent triangles and designate their harmonious qualities by the equal and similar signs $\cong$. In theory of numbers the term congruence still has the same connotation in that groups of numbers have some relation to each other. We are often concerned here with properties of numbers that are true for a whole group of integers. For example, we find a certain property holding for squares of odd integers when divided by 8 leave 1 as a remainder. This property holds for all odd numbers, or a class of numbers differing from each other by multiples of 2. Another example is the fact that when we square any number having 6 as the last digit, the product will also end in 6. Thus, we see another property holding for a whole group of numbers. The consideration of these properties holding for all integers differing from each other by a multiple of a certain integer leads to a notion of congruences.

Carl Friedrich Gauss, who lived in the 18th century, is attributed with the foundation of congruences. He was a mathematical genius, especially with theory of numbers. It is this topic, arithmetical congruences, which he treated so well, that will concern us now. Just what do we mean by arithmetical congruences?

In my paper I will attempt to do three things: (1) show what is meant by two numbers being congruent, (2) show that it is possible to solve for unknowns in linear, quadratic, and cubic congruences, and (3) show how we might apply congruences practically.

What do we mean when we say that two numbers are congruent? Two integers a and b whose difference $a-b$ is divisible by a given number m (not 0) are said to be congruent for the modulus m or simply modulo m. This means simply that two numbers are congruent if they have the same remainder when divided by the same number. This is expressed $a \equiv b$ (mod m) where the congruent sign is written $\equiv$. For example, $17 \equiv 5$ (mod 12) or $9^2 \equiv 1$ (mod 5). Note that the congruence is true either if you substract the integers first or reduce them first. For example,

Substracting first:	Reducing first:
$9^2 - 1 \equiv 0 \pmod 5$	$9^2 \equiv 1 \pmod 5$
	$81/5 \equiv 1/5 \pmod 5$
$81 - 1 \equiv 0 \pmod 5$	$1 \equiv -4 \pmod 5$
$0 \equiv 0 \pmod 5$	$1 \equiv 1 \pmod 5$

Notice also that −4 and +1 are the same remainder modulo 5 as can be shown by dividing 1 by 5 as follows:

$$\begin{array}{r} 1 \\ 5\overline{)1} \\ \underline{5} \\ -4 \end{array} \qquad \begin{array}{r} 0 \\ 5\overline{)1} \\ \underline{0} \\ +1 \end{array}$$

A geometric interpretation of congruences is helpful. The first figure is a representation of integers, and the second figure represents these integers modulo 6.

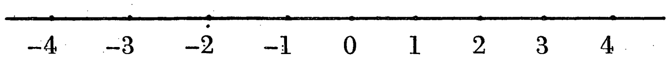


Figure I

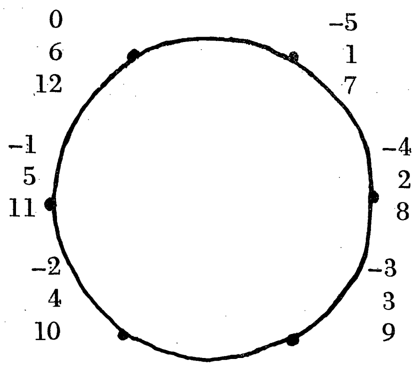
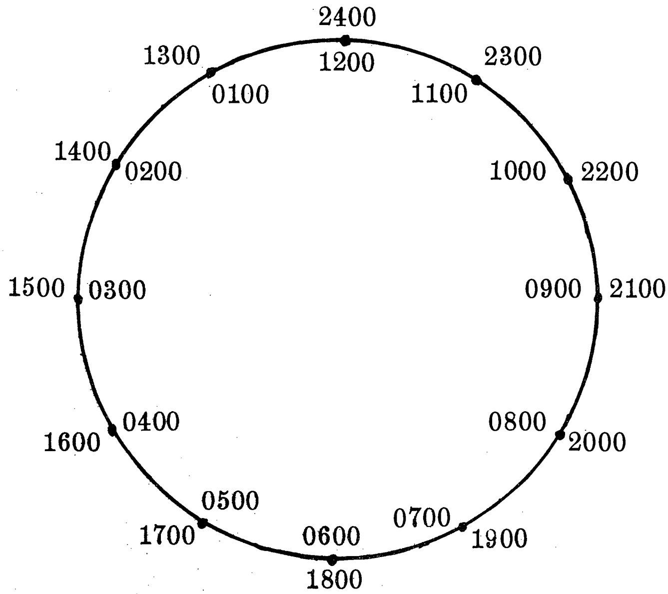


Figure II

Notice that the numbers at any one point in Figure II are congruent with each other modulo 6. For example,

$$\begin{aligned} -5 &\equiv 7 \pmod 6 \\ -5-7 &\equiv 0 \pmod 6 \\ -12 &\equiv 0 \pmod 6 \\ 0 &\equiv 0 \pmod 6 \end{aligned}$$

An application in daily life of this geometric interpretation is the clock. The hands indicate the hour modulo 12. Below is a representation of army time. Whereas we start over with 1:00, 2:00, etc., after reaching 12:00, soldiers begin with 0100 and continue up to 2400 for an entire day.



Notice at each position 12 is added so as to make congruent numbers.

$$0100 \equiv 1300 \pmod{12}$$
$$0100 - 1300 \equiv 0 \pmod{12}$$
$$-1200 \equiv 0 \pmod{12}$$
$$0 \equiv 0 \pmod{12}$$

Many properties of ordinary equations hold for congruences. Some of these are as follows:

a. For any modulus m, $a \equiv a \pmod{m}$.
b. If $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$.
c. If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.
d. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a \equiv c \equiv b \equiv d \pmod{m}$.
e. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.
f. If $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$ where n is a positive integer.
g. A common factor n cannot be removed from $na \equiv nb \pmod{m}$ unless it is a prime of m or unless m is also divided by n.
h. If $a \equiv b \pmod{m}$ and d is a divisor of m, then $a \equiv b \pmod{d}$.
i. If $a \equiv b$ with respect to moduli $m_1, m_2, \cdots m_n$ then $a \equiv b \pmod{l}$ where l is the least common factor of $m_1, m_2, \cdots m_n$.

Let us see how we can apply these principles. Suppose we wanted to find if 47 is a factor of $2^{23}-1$. To obtain 2^{23} would constitute a good bit of work, but with congruences the solution is simple. We simply break down 2^{23} into powers of 2 that we can readily compute, divide each term by 47, multiply the remainders together, again divide by the modulus, and end up with the final remainder. Since the remainder equals 1 we can say that 47 divides $2^{23}-1$ evenly. This problem is shown as follows:

$$2^{23} \equiv 1 \pmod{47}$$
$$2^6 \cdot 2^{12} \cdot 2^5 \equiv 64 \cdot 64^2 \cdot 32 \pmod{47}$$
$$2^6 \cdot 2^{12} \cdot 2^5 \equiv 17 \cdot 7 \cdot 32 \pmod{47}$$
$$2^6 \cdot 2^{12} \cdot 2^5 \equiv 3808 \pmod{47}$$
$$2^{23} \equiv 1 \pmod{47}$$
$$2^{23}-1 \equiv 0 \pmod{47}$$

Arithmetical congruences become more valuable with the solution of linear, quadratic, and cubic congruences with one unknown. Actually, the solutions to each are similar. For example, the linear congruence is solved in the following manner: $2x+1 \equiv 0 \pmod{5}$. The method is simply by trial and error–substituting numbers for x and seeing if they satisfy the congruence. However, there is a limit to the amount of numbers that need to be substituted. Mathematicians have found that only ½ of the positive and negative values of the modulus m need to be tried. Higher numbers, as I have already indicated, can be shown to be the same remainder as one of the lower values. In this case all values up to and including ± 2 need to be tried. We find $+2$ is a solution as well as -3; but $+2$ and -3 is the same remainder modulo 5.

This method holds for the solution of a quadratic also; however, if the congruence can be factored, the work involved is reduced. For example:

$$x^2+5x+6 \equiv 0 \pmod{7}$$
$$(x+3)(x+2) \equiv 0 \pmod{7}$$
$$(x+3) \equiv 0 \pmod{7}$$
$$x \equiv 4 \text{ or } -3 \pmod{7}$$
$$(x+2) \equiv 0 \pmod{7}$$
$$x \equiv 5 \text{ or } -2 \pmod{7}$$

For a cubic congruence such as $x^3+19x^2+x+23 \equiv 0 \pmod{42}$, one can readily see that to substitute in ½ the positive and negative values of the modulus 42 would be quite monotonous, so mathematicians generally employ the Chinese Remainder Theorem to solve the cubic.

Usually the human mind attempts to find some practical application of this knowledge. I have developed an original idea which I think is interesting and perhaps useful to some, in my case to the marching-band director. Let us take a very simple quadratic congruence and illustrate how it might be employed in figuring marching-band formations.

Let us suppose that the college band of St. Benedict's College were marching down Pittsburg's football field. Mr. Roark, St. Benedict's band director, decided to have his band march on the field in a perfect square formation, that is as many rows as columns. Then at the middle of the field he wanted them to go into a P formation for Pittsburg. To make a P he wanted 9 men for the loop and 15 for the stem. He wanted to enclose the P with rows of men, 15 per row. How many men would he need? We could set up the following relation:

$$x^2 - 9 \equiv 0 \pmod{15}$$

$$(x-3)(x+3) \equiv 0 \pmod{15}$$

$$(x-3) \equiv 0 \pmod{15}$$

$$+3 \text{ or } -12 \equiv 0 \pmod{15}$$

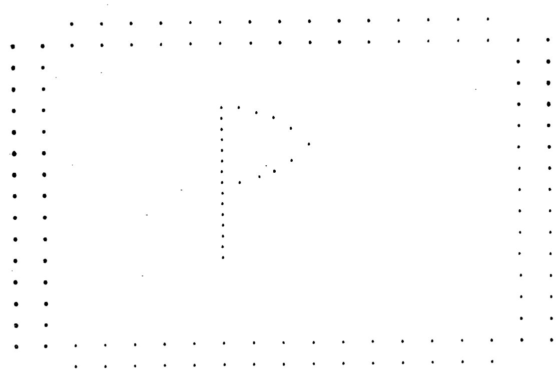
$$(x+3) \equiv 0 \pmod{15}$$

$$-3 \text{ or } +12 \equiv 0 \pmod{15}$$

Using 12 as a solution, we have,

$$144 - 9 \equiv 0 \pmod{15}$$
$$135 \equiv 0 \pmod{15}$$

Now we have 12 rows and 12 columns of men marching down the field. Nine of these will be used for the loop of the P which leaves 135 for the stem and the rows. By dividing 135 by 15 we see that there are 9 rows of 15 men in each. We can therefore set up the following formation:



Perhaps one of the most practical and fascinating applications of congruences is in working calendar problems.

You have heard about people being "walking dictionaries." You can become a "walking calendar." We can figure out the days of the week on which certain dates fall; certain dates of the month on which certain days fall; the day on which you were born; the day of your next birthday; or even dates on which Easter falls. One day in history class when we were discussing the second World War, my professor asked me, "When did the United States declare war on Japan?" I hesitated momentarily, then answered emphatically, "On Monday, December 8, 1941." He looked at me a little surprised to think that I remembered even the day of the week; however, he really would have been surprised if he had known that I didn't remember but that I quickly applied a mathematical formula. This is what I did:

$f \equiv ?$, day $\qquad f \equiv k + (2.6m - 0.2) + D + \frac{D}{4} + \frac{C}{4} - 2C \pmod 7$

$k \equiv 8$, date

$m \equiv 10$, month $\qquad f \equiv 8 + (2.6)(10) - 0.2 + 41 + \frac{41}{4} + \frac{19}{4} - 2(19) \pmod 7$

$D \equiv 41$, years in century

$C \equiv 19$, century

$$f \equiv 8 + 25 + 41 + 10 + 4 - 38 \pmod 7$$

$$f \equiv 88 - 38 \pmod 7$$

$$f \equiv 50 \pmod 7$$

$$f \equiv 1 \pmod 7$$

Since Sunday is 0, Monday is 1, so Monday is the required day of the week.

We could apply congruences in many ways in the home but let's see if we can't apply them in the social world. Let's say you're coming home from the midnight ball with your fiancée. The moon is shining like a silver dollar—and the subject, guess what! Finally you agree—yes, June is the month and Saturdays are the only days you'll consider. "But what date shall we set? Of course you have no calendar with you and at such a vital moment! But, yes, she had studied congruences. She'd figure out on what dates Saturdays fall!

$f \equiv -1$, day $\qquad -k \equiv (2.6m - 0.2) - 2a - 4b + \frac{C}{4} - C - f \pmod 7$

$m \equiv 4$, month

$a \equiv 1$ $\qquad -k \equiv (2.6)(4) - 0.2 - 2(1) - 4(4) + \frac{19}{4} - 19 - (-1)$ mod 7)

$b \equiv 4$

$C \equiv 19$, century $\quad -k \equiv 10 - 2 - 16 + 4 - 19 + 1 \pmod 7$

$k \equiv ?$, date $\qquad -k \equiv -22 \pmod 7$

$$k \equiv 1 \pmod 7$$

Note: a and b are obtained by dividing the year by 4 and 7 respectively and using the remainders. (When using these formulas always use the least whole integer.) She found that in June, 1957, Saturdays fall on the first, the eighth, the fifteenth, the twenty-second, and the twenty-ninth. The remainder 1 equals the first date of the month. Simply add 7 days on to this date each time to find the remaining Saturdays. She applied Zeller's Rule and the wedding date was set!

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What is Mathematics? by Courant and Robbins

The Elements of the Theory of Algebraic Numbers by Legh Wilbur Reid

Elementary Theory of Numbers, Harriet Grittin, pp. 77-80.

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PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

PROPOSALS

334. *Proposed by Huseyin Demir, Kandilli, Eregli, Kdz, Turkey.*

Find the simplest expression for the area S enclosed by the arc AM of a cycloid, the arc TM of the rolling circle Ω (a) and the base line segment AT.

335. *Proposed by Robert E. Shafer, University of California Radiation Laboratory.*

Prove

$$\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i}\binom{n}{i} = \sum_{i=1}^{n} \frac{1}{i}$$

for all $n \geqq 1$.

336. *Proposed by C. W. Trigg, Los Angeles City College.*

A uniform bar with rounded ends and length x is hung from one end by a string of length x and negligible mass. The other end of the string is attached to a vertical wall. When the free end of the bar is placed against the wall, it is found that θ is the smallest angle that can be made between the bar and the wall so that the bar will not slip and fall down. (The plane of the bar and string is perpendicular to the wall.) What is the coefficient of friction between the bar and the wall?

337. *Proposed by Victor Thebault, Tennie, Sarthe, France.*

Determine the right triangles whose sides are integers and whose hypotenuse is twice a square.

338. *Proposed by M. S. Klamkin, AVCO, Lawrence, Massachusetts.*

The vector field $\frac{R}{r^3}$ satisfies the equations $\nabla \times \frac{R}{r^3} = 0$ and $\nabla \cdot \frac{R}{r^3} = 0$. Consequently, this field has a scalar potential and a vector potential. The scalar potential is well known to be $\frac{1}{r}$. Determine the vector potential.

339. *Proposed by D.F. Huntington and D.A. Kearns, University of Maine.*

Criticize the following "proof" that the Cantor Ternary Set (Cantor Discontinuum) is non-denumerable.

At the nth stage of the construction of the set there are 2^n closed intervals, the 2^{n+1} end points of which belong to the set. Since there are a denumerable number of stages in the construction, the total number of end points is $2^{\aleph_0+1} = 2^{\aleph_0} = \aleph_0$. Therefore these end points are non-denumerable, and since they constitute a subset of the Cantor set, the entire Cantor set is non-denumerable.

340. *Proposed by R. G. Buschman, University of Wichita.*

Prove that

$$\sum_{k=2}^{\infty} \frac{(-1)^k(k-1)}{k(k+1)} \zeta(k) = 2\int_0^{\infty} ([e^t] - e^t + 1/2)\, dt$$

where $[x]$ represents the greatest integer not exceeding x and $\zeta(k)$ is the Reimann Zeta-function.

SOLUTIONS

Christmas Sunday

313 [September 1957] *Proposed by Sidney Kravitz, East Paterson, New Jersey.*

Show that Christmas falls on a Sunday more often than once every seven years.

Solution by C. W. Trigg, Los Angeles City College. A usual year contains 365 or 7(52) + 1 days, and every year divisible by 4 contains an extra day unless the year is divisible by 100 and not by 400. It follows that 400 years will include 97 leap years and exactly 146097 days or 20871 weeks. Hence, the calendar repeats every 400 years.

Christmas will fall on Sunday in 1960 and will progress through the seven-day week one day at a time for three days and then will skip a day in leap year. The pattern will repeat every 28 years until broken up by a multiple of 100. For example:

Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
1960	1961	1962	1963		1964	1965
1966	1967		1968	1969	1970	1971
	1972	1973	1974	1975		1976
1977	1978	1979		1980	1981	1982
1983		1984	1985	1986	1987	
. . .	. . .	. . .	. . .	. . .	. . .	. . .
2095		2096	2097	2098	2099	2100
. . .	. . .	. . .	. . .	. . .	. . .	. . .
2196	2197	2198	2199	2200	2201	2202
. . .	. . .	. . .	. . .	. . .	. . .	. . .
2298	2299	2300	2301	2302	2303	

Now Christmas falls on Sunday only once in the span 1961 to 1967, inclusive. Hence, it does not fall on Sunday more often than once "every seven years".

If, however, the statement is interpreted to mean that "the probability of Christmas falling on Sunday is greater than 1/7", then we merely need to consider the days upon which Christmas falls in the 1960-2359 span.

Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
58	56	58	57	57	58	56

Considered in this light the proposition holds.

A similar problem in AMERICAN MATHEMATICAL MONTHLY, **40**, 607, December 1933 shows that the thirteenth is more likely to fall on Friday than on any other day of the week.

Also solved by George Bergman, Stuyvesant High School, New York City; Sam Kravitz, East Cleveland, Ohio; and the proposer, all using the probability approach. D. A. Breault, Sylvania Electric Products, Inc. Waltham, Massachusetts, showed the proposition false between 1967 and 1976.

Powers of Two

314. [September 1957] *Proposed by J.M. Gandhi, Lingraj College, Belgaum, India.*

Show that $2^{n-1} = 2({}_nC_2 - {}_nC_3) + 4({}_nC_4 - {}_nC_5) + 6({}_nC_6 - {}_nC_7) + \cdots$; the last term being $n({}_nC_n)$ when n is even and $-(n-1){}_nC_n$ when n is odd.

I. *Solution by W.B. Carver, Cornell University.* Consider the case when

n is odd. We write C_s for ${}_nC_s$ and Γ_s for ${}_{n-1}C_s$. It is well known that $2^{n-1} = 2\Gamma_1 + 2\Gamma_3 + 2\Gamma_5 + 2\Gamma_7 \cdots + 2\Gamma_{n-2}$. We may write this

$$2^{n-1}=2\Gamma_1+(4-2)\Gamma_3+(6-4)\Gamma_5 \cdots + \{(n-1)-(n-3)\}\Gamma_{n-2}.$$
$$=2(\Gamma_1-\Gamma_3)+4(\Gamma_3-\Gamma_5)+6(\Gamma_5-\Gamma_7)\cdots+(n-3)(\Gamma_{n-4}-\Gamma_{n-2})+(n-1)\Gamma_{n-2}$$
$$=2(\Gamma_1+\Gamma_2-\Gamma_2-\Gamma_3)+4(\Gamma_3+\Gamma_4-\Gamma_4-\Gamma_5)\cdots+(n-3)(\Gamma_{n-4}+\Gamma_{n-3}-\Gamma_{n-3}-\Gamma_{n-2})+ (n-1)\Gamma_{n-2},$$
$$=2(C_2-C_3)+4(C_4-C_5)+6(C_6-C_7)\cdots+(n-3)(C_{n-3}-C_{n-2})+(n-1)\Gamma_{n-2}.$$

Since ${}_{n-2} = C_{n-1}-C_n$, this proves the theorem for n odd. A slight modification of the above takes care of the case n even.

II. *Solution by Michael J. Pascual, Burbank, California.* Consider the function

$$(1) \quad f(x) = -1/x + \frac{(1-x^2)^n}{x}$$

$$f(x) = -1/x + 1/x - {}_nC_1x + {}_nC_2x^3 - {}_nC_3x^5 + \cdots + (-1)^n{}_nC_nx^{2n-1}$$

$$f'(x) = -{}_nC_1 + 3{}_nC_2x^2 - 5{}_nC_3x^4 + \cdots + (-1)^n(2n-1){}_nC_nx^{2n-2}$$

$$f'(x) = -{}_nC_1 + 3{}_nC_2 - 5{}_nC_3 + \cdots + (-1)^n(2n-1){}_nC_n$$

and by differentiating (1) we easily have that $f'(1) = 1 = {}_nC_0$ hence

$${}_nC_0 = -{}_nC_1 + 3{}_nC_2 - 5{}_nC_3 + \cdots + (-1)^n(2n-1){}_nC_n$$

$${}_nC_0+{}_nC_1+{}_nC_2+\cdots+{}_nC_n=4{}_nC_2-4{}_nC_3+8{}_nC_4-8{}_nC_5+\cdots+[(-1)^n(2n-1)+1]\,{}_nC_n$$

but the left member is $(1+1)^n = 2^n$. Hence

$$2^{n-1} = 2({}_nC_2-{}_nC_3)+4({}_nC_4-{}_nC_5)+\cdots+\frac{(-1)^n(2n-1)+1}{2}\,{}_nC_n,$$

and if n is even $\dfrac{(-1)^n(2n-1)+1}{2} = n$

whereas if n is odd $\dfrac{(-1)^n(2n-1)+1}{2} = -(n-1)$ as required.

Also solved by Harry D. Ruderman, New York, New York; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; E. P. Starke, Rutgers University; C. W. Trigg, Los Angeles City College; Chih-yi Wang, University of Minn-

esota and the proposer.

Equivalent Integrals

315. [September 1957] *Proposed by P. D. Thomas, Eglin Air Force Base, Florida.*

Under the restriction $f(c-x) = f(x)$ show that $\int_0^c (a+bx)f(x)\,dx$ may be written $(a+(bc)/2)\int_0^c f(x)\,dx$.

I. *Solution by Ben K. Gold, Los Angeles City College.* Since $f(c-x) = f(x)$ we have

$$\int_0^c [(c-x)+c]f(c-x)[-d(c-x)] = -\int_0^c xf(x)dx + c\int_0^c f(x)dx$$

This leads to

$$\int_0^c xf(x) = \frac{c}{2}\int_0^c f(x)\,dx$$

The result

$$\int_0^c (a+bx)\,f(x)dx = (a+bc/2)\int_0^c f(x)dx$$

follows.

II. *Solution by Waleed A. Al-Salam, Duke University.* Since $f(c-x) = f(x)$, we have $\int_0^c f(x)dx = 2\int_0^{c/2} f(x)\,dx$

Now

$$\int_0^c (a+bx)\,f(x)dx = \int_0^{c/2} (a+bx)\,f(x)dx + \int_{c/2}^c (a+bx)\,f(x)\,dx$$

In the second integral of the right hand side, put $x = c-y$. We get

$$\int_0^c (a+bx)\,f(x)\,dx = \int_0^{c/2} (a+by)\,f(y)\,dy + \int_0^{c/2} (a+b(c-y)\,f(y)\,dy$$

$$= (2a+bc)\int_0^{c/2} f(x)\,dx$$

$$= (a+\frac{bc}{2})\int_0^c f(x)\,dx$$

Also solved by W. B. Carver, Cornell University; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; J. M. Gandhi, Jain Engineering College, Panchkoola, India; Donald A. Kearns, University of Maine; Joseph D. E. Konhauser, Haller, Raymond and Brown, Inc., State College Pennsylvania; M. Morduchow, Polytechnic Institute of Brooklyn; F.D. Parker, University of Alaska; Michael J. Pascual, Burbank, California; Arne Pleyet, Trolhattan, Sweden; A. K. Rajagopal, Indian Institute of Science, Bangalore, India; Lawrence A. Ringenberg, Eastern Illinois State College; Harry D.

Ruderman, New York, New York; William F. Russell, St. Joseph's College, Pennsylvania; Chih-yi Wang, University of Minnesota and the proposer.

Factorization of $1 + \cos n\theta$

316. [September 1957] *Proposed by A. K. Rajagopal, Lingraj College, Belgaum, India.*

Prove that $(1 + \cos n\theta)$, n an integer, has a factor $(1 + \cos\theta)$ if and only if n is odd.

I. *Solution by Joseph D. E. Konhauser, State College, Pennsylvania.* It is well-known that $1 + \cos n\theta$ can be written as a polynomial of degree n in $\cos\theta$. Let $x = \cos\theta$, and denote the polynomial by $p(x)$. The problem is to show that $1 + x$ is a factor of $p(x)$. This is the case if and only if -1 is a zero of $p(x)$, which is equivalent to $\theta = \pi$ being a zero of $1 + \cos n\theta$. But $\theta = \pi$ is a zero of $1 + \cos n\theta$ if and only if n is odd, proving the desired result.

II. *Solution by George Bergman, student at Stuyvesant High School, New York, New York.* Let $x = e^{i\theta}$. Then $(1 + \cos\theta)$ and $(1 + \cos n\theta)$ are equal respectively to $\frac{1}{2}(x + 2 + x^{-1})$ and $\frac{1}{2}(x^n + 2 + x^{-n})$. Now an algebraic factor of such an expression is equivalent to a trigonometric factor if and only if it takes on real values for any x on the unit circle. Since this is the case, we need only test as to whether the first of our algebraic expressions is a factor of the second. The factorization of the first is $\frac{1}{2}(x+1)(x^{-1}+1)$ which the Factor Theorem shows us divides the second only for odd n.

Also solved by Waleed A. Al-Salam, Duke University. Walter B. Carver, Cornell University; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; M. Morduchow, Polytechnic Institute of Brooklyn; F. D. Parker, University of Alaska; Michael J. Pascual, Burbank, California; Arne Pleyet, Trolhattan, Sweden; Lawrence A. Ringenberg, Eastern Illinois University; Harry D. Ruderman, New York, New York; Dmitri Thoro, University of Florida; P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D. C.; Chih-yi Wang, University of Minnesota and the proposer.

An Exponential Inequality

317. [September 1957] *Proposed by Ben K. Gold, Los Angeles City College.*

Prove that $(e + x)^{(e-x)} > (e - x)^{(e+x)}$ for $0 < x < e$.

I. *Solution by E. P. Starke, Rutgers University.* Let

$$f(x) = (e - x)\ln(e+x) - (e + x)\ln(e - x).$$

Then

$$f'(x) = \left(\frac{e+x}{e-x}\right) + \left(\frac{e-x}{e+x}\right) - \ln(e^2 - x^2)$$

Now, remembering that $n + 1/n > 2$ for all positive $n \neq 1$, it is easily seen that $f'(x) > 0$ for $0 < |x| < e$. Hence $f(x)$ is an increasing function over the same range. Thus, since $f(0) = 0$, $(e-x)\ln(e+x) > (e+x)\ln(e-x)$, $0 < x < e$, and this is equivalent to the required result.

II. *Solution by the proposer,* To prove $(e+x)^{(e-x)} > (e-x)^{(e+x)}$ we have the equivalent statement

$$\frac{(e+x)^e}{(e+x)^x} > \frac{(e-x)^e}{(e-x)^x} \quad \text{or} \quad \left(\frac{e+x}{e-x}\right)^e > \left(\frac{e+x}{e-x}\right)^x$$

which is of the form $N^e > N^x$ or $e > x$. But our hypothesis is that $0 < x < e$ so the original statement holds.

Also solved by J. M. Gandhi, Jain Engineering College, Panchkoola, India; Joseph D. E. Konhauser, State College, Pennsylvania; Michael J. Pascual, Burbank, California; A. K. Rajagopal, Indian Institute of Science, Bangalore, India; Harry D. Ruderman, New York, New York; Chih-yi Wang, University of Minnesota.

Limit of a Quotient

318. [September 1957] *Proposed by Chih-yi Wang, University of Minnesota.*

Evaluate $\lim\limits_{x\to\infty} \dfrac{x^{\log x}}{(\log x)^x}$

I. *Solution by D. A. Breault, Sylvania Electric Products, Inc., Waltham, Massachusetts.* Let

$$F(x) = \frac{(x)^{\log x}}{(\log x)^x}, \tag{1}$$

and we wish to find $\lim\limits_{x\to\infty} F(x)$. Set $x = e^u$, and (1) becomes,

$$\lim_{u\to\infty} G(u) = \lim_{u\to\infty}\left[\frac{e^{u^2}}{(u)^{e^u}}\right]. \tag{2}$$

Taking the natural logarithm of both sides of (2), we see that

$$\lim_{u\to\infty} \log G(u) = \lim_{u\to\infty} \{u^2 - e^u \log u\} \tag{3}$$

which clearly is $-\infty$. From this fact, we deduce that $\lim\limits_{u\to\infty} G(u) = 0$, and hence $\lim\limits_{u\to\infty} F(x) = 0$, also.

II. *Solution by M. Morduchow, Polytechnic Institute of Brooklyn.* L'Hopital's Rule will be applied. First the derivative of the numerator and of the denominator must be obtained. To differentiate $y=x^{\log x}$, it is noted that $\log y=(\log x)^2$, and $y'/y=2(\log x)/x$; hence $y'=y[2(\log x)/x]$. Similarly, if $z=(\log x)^x$, then one finds $z'=z[\log(\log x)+(1/\log x)]$. Thus $\lim_{x\to\infty}\frac{y}{z}=\lim_{x\to\infty}\frac{y}{z}\cdot U$, where $U=\dfrac{2(\log x)/x}{\log(\log x)+(1/\log x)}$. Hence $\lim_{x\to\infty}\frac{y}{z}(1-U)=0$. But $U\to 0$ as $x\to\infty$ [as can be readily seen by noting that $(\log x)/x\to 0$]. Hence $y/z\to 0$ as $x\to\infty$.

Also solved by Walter B. Carver, Cornell University; Joseph D. E. Konhauser, State College, Pennsylvania; Michael J. Pascual, Burbank, California; Harry D. Ruderman, New York, New York; E. P. Starke, Rutgers University and the proposer.

A Trigonometric Identity

319. [September 1957] *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania.*

Show that $\dfrac{\sin n\theta}{\sin\theta}=\cos^{n-1}\theta\left\{1+\dfrac{\operatorname{Cos}\theta}{\operatorname{Cos}\theta}+\dfrac{\operatorname{Cos}2\theta}{\operatorname{Cos}^2\theta}+\cdots+\dfrac{\operatorname{Cos}(n-1)\theta}{\operatorname{Cos}^{n-1}\theta}\right\}$ for those values of θ for which the terms are defined.

I. *Solution by Lawrence A. Ringenberg, Eastern Illinois University.* The equation $\dfrac{\sin 2\theta}{\sin\theta}=\cos\theta\left(1+\dfrac{\cos\theta}{\cos\theta}\right)$ follows immediately from the double angle formula $\sin 2\theta=2\sin\theta\cos\theta$. Suppose k is a positive integer such that $\dfrac{\sin k\theta}{\sin\theta}=\cos^{k-1}\theta+\cos^{k-2}\theta\cos\theta+\cdots+\cos\theta\cos(k-2)\theta+\cos(k-1)\theta$.

Then

$$\begin{aligned}\frac{\sin(k+1)\theta}{\sin\theta}&=\frac{\sin(k+1)\theta}{\sin k\theta}\cdot\frac{\sin k\theta}{\sin\theta}\\&=\left(\cos\theta+\frac{\cos k\theta\sin\theta}{\sin k\theta}\right)\frac{\sin k\theta}{\sin\theta}\\&=\cos\theta\,(\cos^{k-1}\theta+\cos^{k-2}\theta\cos\theta+\cdots+\cos(k-1)\theta)+\cos k\theta\\&=\cos^k\theta\left(1+\frac{\cos\theta}{\cos\theta}+\cdots+\frac{\cos k\theta}{\cos^k\theta}\right)\end{aligned}$$

The desired result follows by mathematical induction.

II. *Solution by Dmitri Thoro, University of Florida.* Let $u(x)=\cos x\theta\sec^x\theta$. From the calculus of finite differences it may be readily verified (by differencing) that $\Delta^{-1}u(x)=\cot\theta\sin x\theta\sec^x\theta$. Hence

$$\sum_{x=0}^{n-1} u(x) = \Delta^{-1} u(x)\Big]_{x=0}^{n} = \cot\theta \sin n\theta \sec^n\theta,$$

or

$$\frac{\sin n\theta}{\sin\theta} = \cos^{n-1}\theta\left\{1+\frac{\cos\theta}{\cos\theta}+\frac{\cos 2\theta}{\cos^2\theta}+\cdots+\frac{\cos(n-1)\theta}{\cos^{n-1}\theta}\right\}$$

III. *Solution by Arne Pleyet, Trolhattan, Sweden.* If $R(x)$ indicates the real part of x then we have

$$\cos^{n-1}\theta\left\{1+\frac{\cos\theta}{\cos\theta}+\cdots+\frac{\cos(n-1)\theta}{\cos^{(n-1)}\theta}\right\}=R\cos^{n-1}\theta\left\{1+\frac{e^{i\theta}}{\cos\theta}+\cdots+\frac{e^{i(n-1)\theta}}{\cos^{(n-1)}\theta}\right\}$$

$$=R\cos^{n-1}\theta\left\{\frac{1-\dfrac{e^{in\theta}}{\cos^n\theta}}{1-\dfrac{e^{i\theta}}{\cos\theta}}\right\}$$

$$=R\frac{\cos^n\theta-\cos n\theta-i\sin n\theta}{-i\sin\theta}$$

$$=\frac{\sin n\theta}{\sin\theta}$$

Also solved by D. A. Breault, Waltham, Massachusetts; Brother T. Brendan, St. Mary's College, California; Huseyin Demir, Kandilli, Eregli, Kdz., Turkey; Joseph D. E. Konhauser, State College, Pennsylvania; Michael J. Pascual, Burbank, California; A. K. Rajagopal, Indian Institute of Science, Bangalore, India; Robert J. Wagner (Two solutions), Lebanon Valley College, Pennsylvania; Chih-yi Wang, University of Minnesota and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 220. Show that $\lim_{n\to\infty} \sum_{k=pn}^{qn} 1/k = \log(q/p)$ [*Submitted by Joseph Andrushkiw*]

Q 221. Find $L(x)$ where it is defined by $L(x) = \lim_{n\to\infty} \frac{1}{n}(\sin x \cos\frac{x}{n} + 2\sin\frac{x}{2}\cos\frac{x}{n-1} +$

$\cdots + n\sin\frac{x}{n}\cos x)$. [*Submitted by Barney Bissinger*]

Q 222. On a 26-question test, 5 points were deducted for each wrong answer and 8 points were credited for each correct answer. If all questions were answered, how many were correct if the score was zero? [*Submitted by C. W. Trigg*]

Q 223. Evaluate $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$ [*Submitted by M. S. Klamkin*]

ANSWERS

A 223. Let $x = \Pi - y$ then $I = -\int_{\Pi}^{0} \frac{(\Pi - y)\sin y}{1+\cos^2 y}dy = -\Pi/2\int_{\Pi}^{0} \frac{d(\cos y)}{1+\cos^2 y} = \frac{\Pi^2}{4}$

A 222. The number of answers in each category is inversely proportional to the value, so there were (5/13)(26) or 10 correct answers.

A 221. Let $\lim_{n\to\infty} U_n = U$ and $\lim_{n\to\infty} V_n = V$. Then $\lim_{n\to\infty}\frac{1}{n}(U_1V_n + \cdots + U_nV_1) = UV$.

Take $U_n = n/x \sin x/n$ and $V_n = \cos x/n$ and $L(x) = xUV = x$

$$\lim_{n\to\infty}\sum_{k=pn}^{qn} 1/k = \lim_{n\to\infty}\left[\left(\sum_{k=1}^{qn} 1/k - \log qn\right) - \left(\sum_{k=1}^{pn} 1/k - \log pn\right) - 1/pn + (\log qn - \log pn)\right]$$

$$= C - C - 0 + \log(q/p)$$

A 220. Since $\lim_{n\to\infty}\left(\sum_{k=1}^{n} 1/k - \log n\right) = C$ (Euler's constant), it follows that

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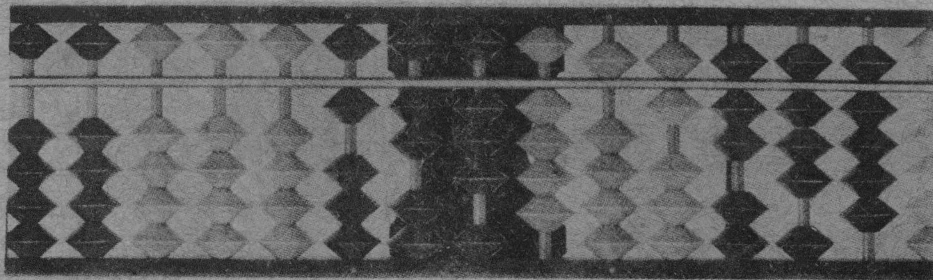

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